# Variable Triebel-Lizorkin-type spaces

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**Abstract**

In this paper we study Triebel-Lizorkin-type spaces with variable smoothness and integrability. We show that our space is well-defined, i.e., independent of the choice of basis functions and we obtain their atomic characterization. Moreover the Sobolev embeddings for these function spaces are obtained.

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## 1 Introduction

Function spaces with variable exponents have been intensively studied in the recent years by a significant number of authors. The motivation to study such function spaces comes from applications to other fields of applied mathematics, such that fluid dynamics and image processing. Some examples of these spaces can be mentioned such as: variable exponent Lebesgue spaces $L^{p(\cdot)}$ and variable Sobolev spaces $W^{k,p(\cdot)}$ where the study of these function spaces was initiated in [23]. Almeida and Samko [2] and Gurka, Harjulehto and Nekvinda [19] introduced variable exponent Bessel potential spaces, wich generalize the classical Bessel potential spaces. Leopold [24-27] and Leopold & Schrohe [28] studied pseudo-differential operators, they introduced related Besov spaces with variable smoothness $B^{\alpha(\cdot)}_{p,p}$. Along a different line of study, J.-S. Xu [49], [50] has studied Besov spaces with variable $p$, but fixed $q$ and $\alpha$. More general function spaces with variable smoothness were explicitly studied by Besov [3], including characterizations by differences. Besov spaces of variable smoothness and integrability, $B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$, initially appeared in the paper of A. Almeida and P. Hästö [1]. Several basic properties were established, such as the Fourier analytical characterisation and Sobolev embeddings. When $p, q, \alpha$ are constants they coincide with the usual function spaces $B^s_{p,q}$. Also, A. I. Tyulenev [40], [41] has studied some new function spaces of variable smoothness.

Triebel-Lizorkin spaces with variable exponents were introduced by [7]. They proved a discretization by the so called $\varphi$-transform. Also atomic and molecular decomposition of these function spaces are obtained and used it to derive trace results. The Sobolev

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embedding of these function spaces was proved by J. Vybíral, [42]. Some properties of these function spaces such as local means characterizations and characterizations by ball means of differences can be found in [21] and [22]. When $\alpha, p, q$ are constants they coincide with the usual function spaces $F^{\alpha}_{p,q}$.

In recent years, there has been increasing interest in a new family of function spaces which generalize the Triebel-Lizorkin spaces, called $F^{s,\tau}_{p,q}$ spaces, were introduced and studied in [43]. When $\tau = 0$, they coincide with the usual function spaces $F^{s}_{p,q}$. Various properties of these function spaces including atomic, molecular or wavelet decompositions, characterizations by differences, have already been established in [10, 12-13, 29, 33-35, 43-46, 48]. Moreover, these function spaces, including some of their special cases related to $Q$ spaces.

Based on Triebel-Lizorkin-type spaces and Triebel-Lizorkin spaces with variable exponents $F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$, we present another Triebel-Lizorkin-type spaces with variable smoothness and integrability, which covers Triebel-Lizorkin-type spaces with fixed exponents. These type of function spaces are introduced in [47], where several properties are obtained such as atomic decomposition and the boundedness of trace operator.

The paper is organized as follows. First we give some preliminaries where we fix some notations and recall some basics facts on function spaces with variable integrability and we give some key technical lemmas needed in the proofs of the main statements. For making the presentation clearer, we give their proofs later in Section 6. In Section 3 we define the spaces $\mathfrak{F}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$ where several basic properties such as the $\varphi$-transform characterization are obtained. In Section 4 we prove elementary embeddings between these functions spaces, as well as Sobolev embeddings. In Section 5, we give the atomic decomposition of these function spaces.

# 2 Preliminaries

As usual, we denote by $\mathbb{R}^n$ the $n$-dimensional real Euclidean space, $\mathbb{N}$ the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The letter $\mathbb{Z}$ stands for the set of all integer numbers. The expression $f \lesssim g$ means that $f \leq c\, g$ for some independent constant $c$ (and non-negative functions $f$ and $g$), and $f \approx g$ means $f \lesssim g \lesssim f$. As usual for any $x \in \mathbb{R}$, $[x]$ stands for the largest integer smaller than or equal to $x$.

By supp $f$ we denote the support of the function $f$ , i.e., the closure of its non-zero set. If $E \subset \mathbb{R}^n$ is a measurable set, then $|E|$ stands for the (Lebesgue) measure of $E$ and $\chi_E$ denotes its characteristic function.

The Hardy-Littlewood maximal operator $\mathcal{M}$ is defined on $L^1_{\mathrm{loc}}$ by

$$\mathcal{M}f(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy$$

and $M_B f(x) = \frac{1}{|B|} \int_B |f(y)|\, dy$, $x \in B$. The symbol $\mathcal{S}(\mathbb{R}^n)$ is used in place of the set of all Schwartz functions $\varphi$ on $\mathbb{R}^n$. We denote by $\mathcal{S}'(\mathbb{R}^n)$ the dual space of all tempered distributions on $\mathbb{R}^n$. The Fourier transform of a tempered distribution $f$ is denoted by $\mathcal{F}f$ while its inverse transform is denoted by $\mathcal{F}^{-1}f$.

For $v \in \mathbb{Z}$ and $m = (m_1, ..., m_n) \in \mathbb{Z}^n$, let $Q_{v,m}$ be the dyadic cube in $\mathbb{R}^n$, $Q_{v,m} = \{(x_1, ..., x_n) : m_i \leq 2^v x_i < m_i + 1, i = 1, 2, ..., n\}$. For the collection of all such cubes we use $\mathcal{Q} := \{Q_{v,m} : v \in \mathbb{Z}, m \in \mathbb{Z}^n\}$. For each cube $Q$, we denote its center by $c_Q$, its lower left-corner by $x_{Q_{v,m}} = 2^{-v}m$ of $Q = Q_{v,m}$ and its side length by $l(Q)$. For $r > 0$, we denote by $rQ$ the cube concentric with $Q$ having the side length $rl(Q)$. Furthermore, we put $v_Q = -\log_2 l(Q)$ and $v_Q^+ = \max(v_Q, 0)$.

For $v \in \mathbb{Z}$, $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, we set $\widetilde{\varphi}(x) := \overline{\varphi(-x)}$, $\varphi_v(x) := 2^{vn}\varphi(2^v x)$, and

$$\varphi_{v,m}(x) := 2^{vn/2}\varphi(2^v x - m) = |Q_{v,m}|^{1/2}\varphi_v(x - x_{Q_{v,m}}) \quad \text{if} \quad Q = Q_{v,m}.$$

The variable exponents that we consider are always measurable functions $p$ on $\mathbb{R}^n$ with range in $[c, \infty[$ for some $c > 0$. We denote the set of such functions by $\mathcal{P}_0$. The subset of variable exponents with range $[1, \infty[$ is denoted by $\mathcal{P}$. We use the standard notation $p^- := \underset{x \in \mathbb{R}^n}{\text{ess-inf}}\, p(x)$, $\quad p^+ := \underset{x \in \mathbb{R}^n}{\text{ess-sup}}\, p(x)$.

The variable exponent Lebesgue space $L^{p(\cdot)}$ is the class of all measurable functions $f$ on $\mathbb{R}^n$ such that the modular

$$\varrho_{p(\cdot)}(f) := \int_{\mathbb{R}^n} |f(x)|^{p(x)}\, dx$$

is finite. This is a quasi-Banach function space equipped with the quasi-norm

$$\|f\|_{p(\cdot)} := \inf\Big\{\mu > 0 : \varrho_{p(\cdot)}\Big(\frac{1}{\mu}f\Big) \leq 1\Big\}.$$

If $p(x) := p$ is constant, then $L^{p(\cdot)} = L^p$ is the classical Lebesgue space.
A useful property is that $\varrho_{p(\cdot)}(f) \leqslant 1$ if and only if $\|f\|_{p(\cdot)} \leqslant 1$ (*unit ball property*). This property is clear for constant exponents due to the obvious relation between the norm and the modular in that case.
Let $p, q \in \mathcal{P}_0$. The space $L^{p(\cdot)}(\ell^{q(\cdot)})$ is defined to be the set of all sequences $(f_v)_{v \geq 0}$ of functions such that

$$\big\|(f_v)_{v\geq 0}\big\|_{L^{p(\cdot)}(\ell^{q(\cdot)})} := \Big\|\big\|(f_v(x))_{v\geq 0}\big\|_{\ell^{q(x)}}\Big\|_{L^{p(\cdot)}} < \infty.$$

It is easy to show that $L^{p(\cdot)}(\ell^{q(\cdot)})$ is always a quasi-normed space and it is a normed space, if $\min(p(x), q(x)) \geq 1$ holds point-wise.
We say that $g : \mathbb{R}^n \to \mathbb{R}$ is *locally* log-*Hölder continuous*, abbreviated $g \in C^{\log}_{\text{loc}}$, if there exists $c_{\log}(g) > 0$ such that

$$|g(x) - g(y)| \leq \frac{c_{\log}(g)}{\log(e + 1/|x - y|)} \tag{1}$$

for all $x, y \in \mathbb{R}^n$. We say that $g$ satisfies the log-*Hölder decay condition*, if there exists $g_\infty \in \mathbb{R}$ and a constant $c_{\log} > 0$ such that

$$|g(x) - g_\infty| \leq \frac{c_{\log}}{\log(e + |x|)}$$

for all $x \in \mathbb{R}^n$. We say that $g$ is *globally*-log-*Hölder continuous*, abbreviated $g \in C^{\log}$, if it is locally log-Hölder continuous and satisfies the log-Hölder decay condition. The

constants $c_{\log}(g)$ and $c_{\log}$ are called the *locally* log-*Hölder constant* and the log-*Hölder decay constant*, respectively. We note that all functions $g \in C^{\log}_{\mathrm{loc}}$ always belong to $L^\infty$.

We define the following class of variable exponents

$$\mathcal{P}^{\log} := \Big\{ p \in \mathcal{P} : \frac{1}{p} \in C^{\log} \Big\},$$

were introduced in [9, Section 2]. We define $1/p_\infty := \lim_{|x|\to\infty} 1/p(x)$ and we use the convention $\frac{1}{\infty} = 0$. Note that although $\frac{1}{p}$ is bounded, the variable exponent $p$ itself can be unbounded. It was shown in [8], Theorem 4.3.8 that $\mathcal{M} : L^{p(\cdot)} \to L^{p(\cdot)}$ is bounded if $p \in \mathcal{P}^{\log}$ and $p^- > 1$, see also [9], Theorem 1.2. Let $p \in \mathcal{P}^{\log}$, $\varphi \in L^1$ and $\Psi(x) := \sup_{|y|\geq|x|} |\varphi(y)|$. We suppose that $\Psi \in L^1$. Then it was proved in [8, Lemma 4.6.3] that

$$\|\varphi_\varepsilon * f\|_{p(\cdot)} \leq c \|\Psi\|_1 \|f\|_{p(\cdot)}$$

for all $f \in L^{p(\cdot)}$, where $\varphi_\varepsilon := \frac{1}{\varepsilon^n}\varphi\left(\frac{\cdot}{\varepsilon}\right)$. We also refer to the papers [5] and [6], where various results on maximal function in variable Lebesgue spaces were obtained.

Recall that $\eta_{v,m}(x) := 2^{nv}(1 + 2^v |x|)^{-m}$, for any $x \in \mathbb{R}^n$, $v \in \mathbb{N}_0$ and $m > 0$. Note that $\eta_{v,m} \in L^1$ when $m > n$ and that $\left\|\eta_{v,m}\right\|_1 = c_m$ is independent of $v$, where this type of function was introduced in [20] and [8]. By $c$ we denote generic positive constants, which may have different values at different occurrences.

## 2.1 Some technical lemmas

In this subsection we present some results which are useful for us. The following lemma is from [22, Lemma 19], see also [7, Lemma 6.1].

**Lemma 1** *Let $\alpha \in C^{\log}_{\mathrm{loc}}$ and let $R \geq c_{\log}(\alpha)$, where $c_{\log}(\alpha)$ is the constant from* (1) *for $\alpha$. Then*

$$2^{v\alpha(x)}\eta_{v,m+R}(x-y) \leq c\ 2^{v\alpha(y)}\eta_{v,m}(x-y)$$

*with $c > 0$ independent of $x, y \in \mathbb{R}^n$ and $v, m \in \mathbb{N}_0$.*

The previous lemma allows us to treat the variable smoothness in many cases as if it were not variable at all, namely we can move the term inside the convolution as follows:

$$2^{v\alpha(x)}\eta_{v,m+R} * f(x) \leq c\ \eta_{v,m} * (2^{v\alpha(\cdot)} f)(x).$$

**Lemma 2** *Let $r, R, N > 0$, $m > n$ and $\theta, \omega \in \mathcal{S}(\mathbb{R}^n)$ with $\operatorname{supp}\mathcal{F}\omega \subset \overline{B(0,1)}$. Then there exists $c = c(r, m, n) > 0$ such that for all $g \in \mathcal{S}'(\mathbb{R}^n)$, we have*

$$|\theta_R * \omega_N * g(x)| \leq c\ \max\Big(1, \Big(\frac{N}{R}\Big)^m\Big)(\eta_{N,m} * |\omega_N * g|^r(x))^{1/r}, \quad x \in \mathbb{R}^n, \qquad (2)$$

*where $\theta_R(\cdot) = R^n\theta(R\cdot)$, $\omega_N(\cdot) = N^n\omega(N\cdot)$ and $\eta_{N,m} := N^n(1 + N|\cdot|)^{-m}$.*

The proof of this lemma is given in [15].

We will make use of the following statement, see [9], Lemma 3.3 and [8], Theorem 3.2.4.

**Theorem 1** *Let $p \in \mathcal{P}^{\log}$. Then for every $m > 0$ there exists $\beta \in (0,1)$ only depending on $m$ and $c_{\log}(p)$ such that*

$$\begin{aligned} &\Big(\frac{\beta}{|Q|}\int_Q |f(y)|\,dy\Big)^{p(x)} \\ \leq\ & \frac{1}{|Q|}\int_Q |f(y)|^{p(y)}\,dy \\ & + \min\left(|Q|^m, 1\right)\Big((e+|x|)^{-m} + \frac{1}{|Q|}\int_Q (e+|y|)^{-m}\,dy\Big), \end{aligned}$$

*for every cube (or ball) $Q \subset \mathbb{R}^n$, all $x \in Q \subset \mathbb{R}^n$ and all $f \in L^{p(\cdot)} + L^\infty$ with $\|f\|_{p(\cdot)} + \|f\|_\infty \leq 1$.*

Notice that in the proof of this theorem we need only

$$\int_Q |f(y)|^{p(y)}\,dy \leq 1$$

and/or $\|f\|_\infty \leq 1$. We set

$$\|(f_v)_v\|_{L^{\tau(\cdot)}_{p(\cdot)}(\ell^{q(\cdot)})} := \sup_{P\in\mathcal{Q}} \Big\|\Big(\frac{f_v}{|P|^{\tau(\cdot)}}\chi_P\Big)_{v\geq v_P^+}\Big\|_{L^{p(\cdot)}(\ell^{q(\cdot)})},$$

where, $v_P = -\log_2 l(P)$ and $v_P^+ = \max(v_P, 0)$. The following result is from [7, Theorem 3.2].

**Theorem 2** *Let $p, q \in C^{\log}_{\text{loc}}$ with $1 < p^- \leq p^+ < \infty$ and $1 < q^- \leq q^+ < \infty$. Then the inequality*

$$\big\|(\eta_{v,m} * f_v)_v\big\|_{L^{p(\cdot)}(\ell^{q(\cdot)})} \leq c\,\|(f_v)_v\|_{L^{p(\cdot)}(\ell^{q(\cdot)})}\,.$$

*holds for every sequence $(f_v)_{v\in\mathbb{N}_0}$ of $L^1_{\text{loc}}$-functions and constant $m > n + c_{\log}(1/q)$.*

We will make use of the following statement (we use it, since the maximal operator is in general not bounded on $L^{p(\cdot)}(\ell^{q(\cdot)})$, see [7, Section 5]).

**Lemma 3** *Let $\tau \in C^{\log}_{\text{loc}}$, $\tau^- > 0$, $p, q \in \mathcal{P}^{\log}$ with $0 < \frac{\tau^+}{\tau^-} < \min(p^-, q^-)$. For $m$ large enough sucth that*

$$m > n\tau^+ + 2n + w,\ \ w > n + c_{\log}(1/q) + c_{\log}(\tau)$$

*and every sequence $(f_v)_{v\in\mathbb{N}_0}$ of $L^1_{\text{loc}}$-functions, there exists $c > 0$ such that*

$$\big\|(\eta_{v,m} * f_v)_v\big\|_{L^{\tau(\cdot)}_{p(\cdot)}(\ell^{q(\cdot)})} \leq c\,\|(f_v)_v\|_{L^{\tau(\cdot)}_{p(\cdot)}(\ell^{q(\cdot)})}\,.$$

The proof of this lemma is postponed to the Appendix.

Let $\widetilde{L^{p(\cdot)}_{\tau(\cdot)}}$ be the collection of functions $f \in L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n)$ such that

$$\|f\|_{\widetilde{L^{p(\cdot)}_{\tau(\cdot)}}} := \sup \Big\|\frac{f\chi_P}{|P|^{\tau(\cdot)}}\Big\|_{p(\cdot)} < \infty, \quad p \in \mathcal{P}_0, \quad \tau : \mathbb{R}^n \to \mathbb{R}^+,$$

where the supremum is taken over all dyadic cubes $P$ with $|P| \geq 1$. Notice that

$$\|f\|_{\widetilde{L^{p(\cdot)}_{\tau(\cdot)}}} \leq 1 \Leftrightarrow \sup_{P\in\mathcal{Q},|P|\geq 1} \Big\|\Big|\frac{f}{|P|^{\tau(\cdot)}}\Big|^{q(\cdot)}\chi_P\Big\|_{p(\cdot)/q(\cdot)} \leq 1. \qquad (3)$$

Let $\theta_v = 2^{vn}\theta(2^v\cdot)$. The following lemma is from [16].

**Lemma 4** *Let $v \in \mathbb{Z}$, $\tau \in C_{\mathrm{loc}}^{\log}$, $\tau^- > 0$, $p \in \mathcal{P}_0^{\log}$, $0 < r < p^-$ and $\theta, \omega \in \mathcal{S}(\mathbb{R}^n)$ with* $\mathrm{supp}\mathcal{F}\omega \subset \overline{B(0,1)}$. *For any $f \in \mathcal{S}'(\mathbb{R}^n)$ and any dyadic cube $P$ with $|P| \geq 1$, we have*

$$\left\| \frac{\theta_v * \omega_v * f}{|P|^{\tau(\cdot)}} \chi_P \right\|_{p(\cdot)} \leq c \left\| \omega_v * f \right\|_{\widetilde{L_{\tau(\cdot)}^{p(\cdot)}}},$$

*such that the right-hand side is finite, where $c > 0$ is independent of $v$ and $l(P)$.*

The next three lemmas are from [7] where the first tells us that in most circumstances two convolutions are as good as one.

**Lemma 5** *For $v_0, v_1 \in \mathbb{N}_0$ and $m > n$, we have*

$$\eta_{v_0,m} * \eta_{v_1,m} \approx \eta_{\min(v_0,v_1),m}$$

*with the constant depending only on $m$ and $n$.*

**Lemma 6** *Let $v \in \mathbb{N}_0$ and $m > n$. Then for any $Q \in \mathcal{Q}$ with $l(Q) = 2^{-v}$, $y \in Q$ and $x \in \mathbb{R}^n$, we have*

$$\eta_{v,m} * \left( \frac{\chi_Q}{|Q|} \right)(x) \approx \eta_{v,m}(x - y)$$

*with the constant depending only on $m$ and $n$.*

**Lemma 7** *Let $v, j \in \mathbb{N}_0$, $r \in (0,1]$ and $m > \frac{n}{r}$. Then for any $Q \in \mathcal{Q}$ with $l(Q) = 2^{-v}$, we have*

$$(\eta_{j,m} * \eta_{v,m} * \chi_Q)^r \approx 2^{(v-j)^+ n(1-r)} \eta_{j,mr} * \eta_{v,mr} * \chi_Q,$$

*where the constant depends only on $m$, $n$ and $r$.*

The next lemma is a Hardy-type inequality which is easy to prove.

**Lemma 8** *Let $0 < a < 1$, $J \in \mathbb{Z}$ and $0 < q \leq \infty$. Let $(\varepsilon_k)_k$ be a sequences of positive real numbers and denote $\delta_k = \sum_{j=J^+}^{k} a^{k-j} \varepsilon_j$, $k \geq J^+$. Then there exists constant $c > 0$ depending only on $a$ and $q$ such that*

$$\Big( \sum_{k=J^+}^{\infty} \delta_k^q \Big)^{1/q} \leq c \Big( \sum_{k=J^+}^{\infty} \varepsilon_k^q \Big)^{1/q}.$$

**Lemma 9** *Let $\alpha, \tau \in C_{\mathrm{loc}}^{\log}$, $\tau^- \geq 0$ and $p, q \in \mathcal{P}_0^{\log}$ with $0 < q^- \leq q^+ < \infty$. Let $(f_k)_{k \in \mathbb{N}_0}$ be a sequence of measurable functions on $\mathbb{R}^n$. For all $v \in \mathbb{N}_0$ and $x \in \mathbb{R}^n$, let $g_v(x) = \sum_{k=0}^{\infty} 2^{-|k-v|\delta} f_k(x)$. Then there exists a positive constant $c$, independent of $(f_k)_{k \in \mathbb{N}_0}$ such that*

$$\left\| (g_v)_v \right\|_{L_{p(\cdot)}^{\tau(\cdot)}(\ell^{q(\cdot)})} \leq c \left\| (f_v)_v \right\|_{L_{p(\cdot)}^{\tau(\cdot)}(\ell^{q(\cdot)})}, \qquad \delta > \tau^+.$$

The proof of Lemma 9 can be obtained by the same arguments used in [15, Lemma 2.10].
The following statement can be found in [4], that plays an essential role later on.

**Lemma 10** *Let real numbers $s_1 < s_0$ be given, and $\sigma \in ]0,1[$. For $0 < q \leq \infty$ and $J \in \mathbb{N}_0$ there is $c > 0$ such that*

$$\left\| (2^{(\sigma s_0 + (1-\sigma)s_1)j} a_j)_{j \geq J} \right\|_{\ell^q} \leq c \left\| (2^{s_0 j} a_j)_{j \geq J} \right\|_{\ell^\infty}^{\sigma} \left\| (2^{s_1 j} a_j)_{j \geq J} \right\|_{\ell^\infty}^{1-\sigma}$$

*holds for all complex sequences $(2^{s_0 j} a_j)_{j \in \mathbb{N}_0}$ in $\ell^\infty$.*

## 3 The spaces $\mathfrak{F}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}$

In this section we present the Fourier analytical definition of Triebel-Lizorkin-type spaces of variable smoothness and integrability and we prove the basic properties in analogy to the Triebel-Lizorkin-type spaces with fixed exponents. Select a pair of Schwartz functions $\Phi$ and $\varphi$ satisfy

$$\operatorname{supp}\mathcal{F}\Phi \subset \overline{B(0,2)} \text{ and } |\mathcal{F}\Phi(\xi)| \geq c \text{ if } |\xi| \leq \frac{5}{3} \tag{4}$$

and

$$\operatorname{supp}\mathcal{F}\varphi \subset \overline{B(0,2)}\backslash B(0,1/2) \text{ and } |\mathcal{F}\varphi(\xi)| \geq c \text{ if } \frac{3}{5} \leq |\xi| \leq \frac{5}{3} \tag{5}$$

where $c > 0$. Let $\alpha : \mathbb{R}^n \to \mathbb{R}$, $p, q, \tau \in \mathcal{P}_0$ and $\Phi$ and $\varphi$ satisfy (4) and (5), respectively and we put $\varphi_v = 2^{vn}\varphi(2^v \cdot)$.

**Definition 1** *Let $\alpha : \mathbb{R}^n \to \mathbb{R}$, $\tau : \mathbb{R}^n \to \mathbb{R}^+$, $p, q \in \mathcal{P}_0$ and $\Phi$ and $\varphi$ satisfy* (4) *and* (5)*, respectively and we put $\varphi_v = 2^{vn}\varphi(2^v \cdot)$. The Triebel-Lizorkin-type space $\mathfrak{F}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that*

$$\|f\|_{\mathfrak{F}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}} := \sup_{P\in\mathcal{Q}} \Big\| \Big( \frac{2^{v\alpha(\cdot)}\varphi_v * f}{|P|^{\tau(\cdot)}} \chi_P \Big)_{v \geq v_P^+} \Big\|_{L^{p(\cdot)}(\ell^{q(\cdot)})} < \infty, \tag{6}$$

*where $\varphi_0$ is replaced by $\Phi$.*

Using the system $(\varphi_v)_{v\in\mathbb{N}_0}$ we can define the norm

$$\|f\|_{F_{p,q}^{\alpha,\tau}} := \sup_{P\in\mathcal{Q}} \frac{1}{|P|^{\tau}} \Big\| \Big( \sum_{v=v_P^+}^{\infty} 2^{v\alpha q} |\varphi_v * f|^q \chi_P \Big)^{1/q} \Big\|_p$$

for constants $\alpha$ and $p, q \in (0, \infty]$. The Triebel-Lizorkin-type space $F_{p,q}^{\alpha,\tau}$ consist of all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ for which $\|f\|_{F_{p,q}^{\alpha,\tau}} < \infty$. It is well-known that these spaces do not depend on the choice of the system $(\varphi_v)_{v\in\mathbb{N}_0}$ (up to equivalence of quasinorms). Further details on the classical theory of these spaces can be found in [10] and [48]; see also [13] for recent developments.

One recognizes immediately that if $\alpha$, $\tau$, $p$ and $q$ are constants, then $\mathfrak{F}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)} = F_{p,q}^{\alpha,\tau}$. When, $q := \infty$ the Triebel-Lizorkin-type space $\mathfrak{F}_{p(\cdot),\infty}^{\alpha(\cdot),\tau(\cdot)}$ consist of all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\Big\| \sup_{P\in\mathcal{Q}, v \geq v_P^+} \frac{2^{v\alpha(\cdot)}\varphi_v * f}{|P|^{\tau(\cdot)}} \chi_P \Big\|_{p(\cdot)} < \infty.$$

Let $B_J$ be any ball of $\mathbb{R}^n$ with radius $2^{-J}$, $J \in \mathbb{Z}$. In the definition of the spaces $\mathfrak{F}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}$ if we replace the dyadic cubes $P$ by the balls $B_J$, then we obtain equivalent quasi-norms. From these if we replace dyadic cubes $P$ in Definition 1 by arbitrary cubes $P$, we then obtain equivalent quasi-norms.

The Triebel-Lizorkin space of variable smoothness and integrability $F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} := \left\| \left( 2^{v\alpha(\cdot)} \varphi_v * f \right)_{v\geq 0} \right\|_{L^{p(\cdot)}(\ell^{q(\cdot)})} < \infty,$$

which introduced and investigated in [7], see [22] and [30] for further results. Obviously, $\mathfrak{F}^{\alpha(\cdot),0}_{p(\cdot),q(\cdot)} = F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$. We refer the reader to the recent paper [46] for further details, historical remarks and more references on embeddings of Triebel-Lizorkin-type spaces with fixed exponents. More information about Triebel-Lizorkin spaces can be found in [35-38].

Sometimes it is of great service if one can restrict $\sup_{P\in\mathcal{Q}}$ in the definition to a supremum taken with respect to dyadic cubes with side length $\leq 1$.

**Lemma 11** *Let $\alpha, \tau \in C^{\log}_{\mathrm{loc}}$, $\tau^- \geq 0$ and $p, q \in \mathcal{P}^{\log}_0$ with $\left(\tau - \frac{1}{p}\right)^- \geq 0$ and $0 < q^+, p^+ < \infty$. A tempered distribution $f$ belongs to $\mathfrak{F}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$ if and only if,*

$$\|f\|^{\#}_{\mathfrak{F}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}} := \sup_{P\in\mathcal{Q},|P|\leq 1} \left\| \left( \frac{2^{v\alpha(\cdot)}\varphi_v * f}{|P|^{\tau(\cdot)}} \chi_P \right)_{v\geq v_P} \right\|_{L^{p(\cdot)}(\ell^{q(\cdot)})} < \infty.$$

*Furthermore, the quasi-norms $\|f\|_{\mathfrak{F}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}}$ and $\|f\|^{\#}_{\mathfrak{F}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}}$ are equivalent.*

**Proof.** Let $P$ be a dyadic cube such that $|P| = 2^{-Jn}$, for some $-J \in \mathbb{N}$. Let $\{Q_m : m = 1, ..., 2^{-Jn}\}$ be the collection of all dyadic cubes with volume 1 and such that $P = \cup_{m=1}^{2^{-Jn}} Q_m$. Due to the homogeneity, we may assume that $\|f\|^{\#}_{\mathfrak{F}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}} = 1$. It suffices to show that

$$\int_P \Big( \sum_{v=v_P^+}^{\infty} 2^{v\alpha(x)q(x)} \frac{|\varphi_v * f(x)|^{q(x)}}{|P|^{\tau(x)q(x)}} \Big)^{p(x)/q(x)} dx \lesssim 1$$

for any dyadic cube $P$, with $|P| \geq 1$. The left-hand side can be rewritten us

$$\begin{aligned} & \sum_{m=1}^{2^{-Jn}} \int_{Q_m} \Big( \sum_{v=v_P^+}^{\infty} 2^{v\alpha(x)q(x)} \frac{|\varphi_v * f(x)|^{q(x)}}{|P|^{\tau(x)q(x)}} \Big)^{p(x)/q(x)} dx \\ \leq \ & 2^{Jn} \sum_{m=1}^{2^{-Jn}} \int_{Q_m} \Big( \sum_{v=v_{Q_m}^+}^{\infty} 2^{v\alpha(x)q(x)} \frac{|\varphi_v * f(x)|^{q(x)}}{|Q_m|^{\tau(x)q(x)}} \Big)^{p(x)/q(x)} dx \\ \lesssim \ & 2^{Jn} \sum_{m=1}^{2^{-Jn}} 1 = 1, \end{aligned}$$

where we used $|P|^{p(x)\tau(x)} \geq 2^{-Jn}$, which completes the proof. ■

**Remark 1** *We like to point out that this result with fixed exponents is given in [48, Lemma 2.2].*

**Theorem 3** *Let $\alpha, \tau \in C^{\log}_{\rm loc}$, $\tau^- \geq 0$ and $p, q \in \mathcal{P}^{\log}_0$ with $0 < q^+, p^+ < \infty$. If $\left(\tau - \frac{1}{p}\right)^- > 0$ or $\left(\tau - \frac{1}{p}\right)^- \geq 0$ and $q := \infty$, then*

$$\mathfrak{F}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)} = B^{\alpha(\cdot)+n(\tau(\cdot)-1/p(\cdot))}_{\infty,\infty}$$

*with equivalent quasi-norms.*

**Proof.** We consider only $\left(\tau - \frac{1}{p}\right)^- > 0$. The case $\left(\tau - \frac{1}{p}\right)^- \geq 0$ and $q := \infty$ can be proved analogously with the necessary modifications. Since $\left(\tau - \frac{1}{p}\right)^- > 0$, then we use the equivalent norm given in the previous lemma. First let us prove the following estimate

$$\|f\|_{\mathfrak{F}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}} \lesssim \|f\|_{B^{\alpha(\cdot)+n(\tau(\cdot)-1/p(\cdot))}_{\infty,\infty}}$$

for any $f \in B^{\alpha(\cdot)+n(\tau(\cdot)-1/p(\cdot))}_{\infty,\infty}$. Let $P$ be a dyadic cube with volume $2^{-nv_P}$, $v_P \in \mathbb{N}$. We obtain that

$$\begin{aligned}
\frac{2^{v\alpha(x)}|\varphi_v * f(x)|}{|P|^{\tau(x)}} &\leq c\, 2^{v(\alpha(x)+n(\tau(x)-1/p(x))+n(v_P-v)(\tau(x)-1/p(x))+nv_P/p(x)}|\varphi_v * f(x)| \\
&\leq c\, 2^{n(v_P-v)(\tau(x)-1/p(x))+nv_P/p(x)}\, \|f\|_{B^{\alpha(\cdot)+n(\tau(\cdot)-1/p(\cdot))}_{\infty,\infty}}
\end{aligned}$$

for any $x \in P$. Then for any $v \geq v_P$

$$\begin{aligned}
&\Big\|\Big(\sum_{v=v_P}^{\infty}\Big|\frac{2^{v\alpha(\cdot)}\varphi_v * f}{|P|^{\tau(\cdot)}}\Big|^{q(\cdot)}\chi_P\Big)^{1/q(\cdot)}\Big\|_{p(\cdot)} \\
\lesssim\ & \|f\|_{B^{\alpha(\cdot)+n(\tau(\cdot)-1/p(\cdot))}_{\infty,\infty}}\Big\|\Big(\sum_{v=v_P}^{\infty}\big|2^{n(v_P-v)(\tau(\cdot)-1/p(\cdot))+nv_P/p(\cdot)}\big|^{q(\cdot)}\Big)^{1/q(\cdot)}\chi_P\Big\|_{p(\cdot)} \\
\lesssim\ & c\ \|f\|_{B^{\alpha(\cdot)+n(\tau(\cdot)-1/p(\cdot))}_{\infty,\infty}}\big\|2^{nv_P/p(\cdot)}\chi_P\big\|_{\frac{p(\cdot)}{q(\cdot)}} \lesssim 1,
\end{aligned}$$

since $\left(\tau - \frac{1}{p}\right)^- > 0$. Let $f \in \mathfrak{F}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$, with $\|f\|_{\mathfrak{F}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}} = 1$. By Lemma 2 we have for any $x \in \mathbb{R}^n, m > n$

$$\begin{aligned}
& 2^{v(\alpha(x)+n(\tau(x)-1/p(x))}|\varphi_v * f(x)| \\
\leq\ & c\, 2^{v(\alpha(x)+n(\tau(x)-1/p(x))}(\eta_{v,m} * |\varphi_v * f|^{p^-}(x))^{1/p^-} \\
\leq\ & c\left\|2^{v(\alpha(x)+n\tau(x))}\varphi_v * f(\cdot)(1+2^v|x-\cdot|)^{-m/2p^-}\right\|_{p(\cdot)}\left\|2^{vn/t(x)}(1+2^v|x-\cdot|)^{-m/2p^-}\right\|_{t(\cdot)},
\end{aligned}$$

by Hölder's inequality, with $\frac{1}{p^-} = \frac{1}{p(\cdot)} + \frac{1}{t(\cdot)}$. The second norm on the right-hand side is bounded if $m > \frac{2p^-}{t^-}(n + c_{\log}(1/t))$ (this is possible since $m$ can be taken large enough). To show that, we investigate the corresponding modular:

$$\begin{aligned}
\varrho_{t(\cdot)}(2^{vn/t(x)}(1+2^v|x-\cdot|)^{-m/2p^-}) &= \int_{\mathbb{R}^n} 2^{vnt(y)/t(x)}(1+2^v|x-y|)^{-mt(y)/2p^-}dy \\
&\leq 2^{vn}\int_{\mathbb{R}^n}(1+2^v|x-y|)^{-(m-c_{\log}(1/t))t^-/2p^-}dy < \infty,
\end{aligned}$$

where we used Lemma 1. Again by the same lemma the first norm is bounded by

$$\left\|2^{v(\alpha(\cdot)+n\tau(\cdot))}\varphi_v * f(\cdot)(1+2^v|x-\cdot|)^{-h}\right\|_{p(\cdot)},$$

where $h = \frac{m}{2p^-} - c_{\log}(\alpha + n\tau)$. Let now prove that this expression is bounded. We investigate the corresponding modular:

$$\begin{aligned}
&\varrho_{p(\cdot)}(2^{v(\alpha(\cdot)+n\tau(\cdot))}\varphi_v * f(\cdot)(1+2^v|x-\cdot|)^{-h}) \\
&= \int_{\mathbb{R}^n} 2^{v(\alpha(y)+n\tau(y))p(y)}|\varphi_v * f(y)|^{p(y)}(1+2^v|x-y|)^{-hp(y)}dy \\
&= \int_{|y-x|<2^{-v}} (\cdots)dy + \sum_{i=1}^{\infty} \int_{2^{i-v}\leq|y-x|<2^{i-v+1}} (\cdots)dy \\
&\leq \sum_{i=0}^{\infty} 2^{-ihp^-} \int_{|y-x|<2^{i-v+1}} 2^{v(\alpha(y)+n\tau(y))p(y)}|\varphi_v * f(y)|^{p(y)}dy. \qquad (7)
\end{aligned}$$

Then the right-hand side of (7) is bounded by

$$\begin{aligned}
&\sum_{i=0}^{\infty} 2^{(n\tau^+p^+-hp^-)i} \int_{B(x,2^{i-v+1})} \frac{2^{v\alpha(y)p(y)}|\varphi_v * f(y)|^{p(y)}}{|B(x,2^{i-v+1})|^{p(y)\tau(y)}}dy \\
\leq\ &\sum_{i=0}^{\infty} 2^{(n\tau^+p^+-hp^-)i} \int_{B(x,2^{i-v+1})} \Big(\sum_{j=(v-i-1)^+}^{\infty} \Big(\frac{2^{j\alpha(y)q(y)}|\varphi_j * f(y)|^{q(y)}}{|B(x,2^{i-v+1})|^{q(y)\tau(y)}}\Big)\Big)^{p(y)/q(y)}dy
\end{aligned}$$

and since $\|f\|_{\mathfrak{F}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}} = 1$, the lat term is bounded by

$$C\sum_{i=0}^{\infty} 2^{(n\tau^+p^+-hp^-)i} < \infty$$

for any $h > \tau^+p^+/p^-$. The proof is completed by the scaling argument. ■

**Remark 2** From this theorem we obtain

$$2^{v(\alpha(x)+n(\tau(x)-1/p(x))}|\varphi_v * f(x)| \leq c\,\|f\|_{\mathfrak{F}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}} \qquad (8)$$

for any $f \in \mathfrak{F}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}$, $x \in \mathbb{R}^n$, $\alpha, \tau \in C_{\mathrm{loc}}^{\log}$, $\tau^- \geq 0$ and $p, q \in \mathcal{P}_0^{\log}$.

In the following theorem we prove the possibility to define these spaces by replacing $v \geq v_P^+$ by $v \geq 0$, in Definition 1. For fixed exponents, see [34].

**Theorem 4** *Let $\alpha, \tau \in C_{\mathrm{loc}}^{\log}$, $\tau^- \geq 0$ and $p, q \in \mathcal{P}_0^{\log}$ with $0 < q^+, p^+ < \infty$. If $\left(\tau - \frac{1}{p}\right)^+ < 0$ or $\left(\tau - \frac{1}{p}\right)^+ \leq 0$ and $q := \infty$, then*

$$\|f\|^*_{\mathfrak{F}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}} = \sup_{P\in\mathcal{Q}} \left\|\left(\frac{2^{\alpha(\cdot)}\varphi_v * f}{|P|^{\tau(\cdot)}}\chi_P\right)_{v\geq 0}\right\|_{L^{p(\cdot)}(\ell^{q(\cdot)})},$$

*is an equivalent quasi-norm in $\mathfrak{F}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}$.*

**Proof.** Clearly, it suffices to prove that $\|f\|^*_{\mathfrak{F}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}} \lesssim \|f\|_{\mathfrak{F}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}}$. In view of the proof of previous theorem, we have

$$2^{v(\alpha(x)+n(\tau(x)-1/p(x))}|\varphi_v * f(x)| \le c\, \|f\|_{\mathfrak{F}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}}$$

for any $x \in \mathbb{R}^n$. Then

$$\begin{aligned} & \Big\| \Big( \sum_{v=0}^{v_P} \Big| \frac{2^{v\alpha(\cdot)}\varphi_v * f}{|P|^{\tau(\cdot)}} \Big|^{q(\cdot)} \Big)^{1/q(\cdot)} \chi_P \Big\|_{p(\cdot)} \\ \lesssim\ & \|f\|_{\mathfrak{F}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}} \Big\| \Big( \sum_{v=0}^{v_P} \big| 2^{(v_P - v)(\tau(\cdot)-1/p(\cdot))+nv_P/p(\cdot)} \big|^{q(\cdot)} \Big)^{1/q(\cdot)} \chi_P \Big\|_{p(\cdot)} \\ \lesssim\ & \|f\|_{\mathfrak{F}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}} \Big\| \big|2^{nv_P/p(\cdot)}\big|^{q(\cdot)} \chi_P \Big\|_{\frac{p(\cdot)}{q(\cdot)}} \lesssim 1, \end{aligned}$$

since $\left(\tau - \frac{1}{p}\right)^+ < 0$. The case $q := \infty$ can be easily solved. ∎

Let $\Phi$ and $\varphi$ satisfy, respectively (4) and (5). By [18, pp. 130–131], there exist functions $\Psi \in \mathcal{S}(\mathbb{R}^n)$ satisfying (4) and $\psi \in \mathcal{S}(\mathbb{R}^n)$ satisfying (5) such that for all $\xi \in \mathbb{R}^n$

$$\mathcal{F}\widetilde{\Phi}(\xi)\mathcal{F}\Psi(\xi) + \sum_{j=1}^{\infty} \mathcal{F}\widetilde{\varphi}(2^{-j}\xi)\mathcal{F}\psi(2^{-j}\xi) = 1, \quad \xi \in \mathbb{R}^n. \tag{9}$$

Furthermore, we have the following identity for all $f \in \mathcal{S}'(\mathbb{R}^n)$; see [18, (12.4)]

$$\begin{aligned} f &= \Psi * \widetilde{\Phi} * f + \sum_{v=1}^{\infty} \psi_v * \widetilde{\varphi}_v * f \\ &= \sum_{m\in\mathbb{Z}^n} \widetilde{\Phi} * f(m)\Psi(\cdot - m) + \sum_{v=1}^{\infty} 2^{-vn} \sum_{m\in\mathbb{Z}^n} \widetilde{\varphi}_v * f(2^{-v}m)\psi_v(\cdot - 2^{-v}m). \end{aligned}$$

Recall that the $\varphi$-transform $S_\varphi$ is defined by setting $(S_\varphi)_{0,m} = \langle f, \Phi_m\rangle$ where $\Phi_m(x) = \Phi(x-m)$ and $(S_\varphi)_{v,m} = \langle f, \varphi_{v,m}\rangle$ where $\varphi_{v,m}(x) = 2^{vn/2}\varphi(2^v x - m)$ and $v \in \mathbb{N}$. The inverse $\varphi$-transform $T_\psi$ is defined by

$$T_\psi \lambda = \sum_{m\in\mathbb{Z}^n} \lambda_{0,m}\Psi_m + \sum_{v=1}^{\infty}\sum_{m\in\mathbb{Z}^n} \lambda_{v,m}\psi_{v,m},$$

where $\lambda = \{\lambda_{v,m} \in \mathbb{C} : v \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$, see [18].

For any $\gamma \in \mathbb{Z}$, we put

$$\|f\|^*_{\mathfrak{F}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}} := \sup_{P\in\mathcal{Q}} \Big\| \Big( \frac{2^{v\alpha(\cdot)}\varphi_v * f}{|P|^{\tau(\cdot)}} \chi_P \Big)_{v \ge v_P^+ - \gamma} \Big\|_{L^{p(\cdot)}(\ell^{q(\cdot)})} < \infty$$

where $\varphi_{-\gamma}$ is replaced by $\Phi_{-\gamma}$.

**Lemma 12** *Let $\alpha, \tau \in C^{\log}_{\mathrm{loc}}$, $\tau^- > 0$, $p, q \in \mathcal{P}^{\log}_0$ and $0 < q^+, p^+ < \infty$. The quasi-norms $\|f\|^*_{\mathfrak{F}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}}$ and $\|f\|_{\mathfrak{F}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}}$ are equivalent with equivalent constants depending on $\gamma$.*

**Proof.** The proof follows the ideas in [48] and [15]. By similarity, we only consider the case $\gamma > 0$. First let us prove that $\|f\|^{*}_{\mathfrak{F}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}} \leq c\,\|f\|_{\mathfrak{F}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}}$. By the scaling argument, it suffices to consider the case $\|f\|_{\mathfrak{F}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}} = 1$ and show that the modular of $f$ on the left-hand side is bounded. In particular, we will show that

$$\Big\|\Big(\sum_{v=v_P^+-\gamma}^{\infty}\Big|\frac{2^{v\alpha(\cdot)}\varphi_v * f}{|P|^{\tau(\cdot)}}\Big|^{q(\cdot)}\Big)^{1/q(\cdot)}\chi_P\Big\|_{p(\cdot)} \leq c$$

for any dyadic cube $P$. We will use the same arguments of [15]. As in [48, Lemma 2.6], it suffices to prove that for all dyadic cube $P$ with $l(P) \geq 1$,

$$I_P = \Big\|\Big(\sum_{v=-\gamma}^{0}\Big|\frac{2^{v\alpha(\cdot)}\varphi_v * f}{|P|^{\tau(\cdot)}}\Big|^{q(\cdot)}\Big)^{1/q(\cdot)}\chi_P\Big\|_{p(\cdot)} \leq c$$

and for all dyadic cube $P$ with $l(P) < 1$,

$$J_P = \Big\|\Big(\sum_{v=v_P-\gamma}^{v_P-1}\Big|\frac{2^{v\alpha(\cdot)}\varphi_v * f}{|P|^{\tau(\cdot)}}\Big|^{q(\cdot)}\Big)^{1/q(\cdot)}\chi_P\Big\|_{p(\cdot)} \leq c.$$

The estimate of $I_P$, clearly follows from the inequality $\Big\|\frac{\varphi_v * f}{|P|^{\tau(\cdot)}}\chi_P\Big\|_{p(\cdot)} \leq c$ for any $v = -\gamma, ..., 0$ and any dyadic cube $P$ with $l(P) \geq 1$. By (4) and (5), there exist $\omega_v \in \mathcal{S}(\mathbb{R}^n)$, $v = -\gamma, \cdot\cdot\cdot, -1$ and $\eta_1, \eta_2 \in \mathcal{S}(\mathbb{R}^n)$ such that

$$\varphi_v = \omega_v * \Phi, \quad v = -\gamma, \cdot\cdot\cdot, -1 \quad \text{and} \quad \varphi = \varphi_0 = \eta_1 * \Phi + \eta_2 * \varphi_1.$$

Hence $\varphi_v * f = \omega_v * \Phi * f$ for $v = -\gamma, ..., -1$ and $\varphi_0 * f = \eta_1 * \Phi * f + \eta_2 * \varphi_1 * f$. Applying Lemma 4, (3) and the fact that $\|f\|_{\mathfrak{F}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}} \leq 1$ to estimate $\Big\|\frac{\varphi_v * f}{|P|^{\tau(\cdot)}}\chi_P\Big\|_{p(\cdot)}$ by

$$C\,\|\Phi * f\|_{\widetilde{L^{p(\cdot)}_{\tau(\cdot)}}} + C\,\|\varphi_1 * f\|_{\widetilde{L^{p(\cdot)}_{\tau(\cdot)}}} \leq c.$$

To estimate $J_P$, denote by $P(\gamma)$ the dyadic cube containing $P$ with $l(P(\gamma)) = 2^{\gamma} l(P)$. If $v_P \geq \gamma + 1$, applying the fact that $v_{P(\gamma)} = v_P - \gamma$ and $P \subset P(\gamma)$, we then have

$$J_P \lesssim \Big\|\Big(\sum_{v=v_{P(\gamma)}}^{v_P-1}\Big|\frac{2^{v\alpha(\cdot)}\varphi_v * f}{|P(\gamma)|^{\tau(\cdot)}}\Big|^{q(\cdot)}\Big)^{1/q(\cdot)}\chi_{P(\gamma)}\Big\|_{p(\cdot)} \leq c.$$

If $1 \leq v_P \leq \gamma$, we write $J_P = \sum_{v=v_P-\gamma}^{-1} \cdots + \sum_{v=0}^{v_P-1} \cdots = J_P^1 + J_P^2$. Let $P(2^{v_P})$ the dyadic cube containing $P$ with $l(P(2^{v_P})) = 2^{v_P} l(P) = 1$. By the fact that $\frac{|P(2^{v_P})|^{\tau(\cdot)}}{|P|^{\tau(\cdot)}} \lesssim 2^{nv_P\tau^+} \lesssim c(\gamma)$ we have

$$J_P^2 \lesssim \Big\|\Big(\sum_{v=v_{P(2^{v_P})}}^{v_P-1}\Big|\frac{2^{v\alpha(\cdot)}\varphi_v * f}{|P(2^{v_P})|^{\tau(\cdot)}}\Big|^{q(\cdot)}\Big)^{1/q(\cdot)}\chi_{P(2^{v_P})}\Big\|_{p(\cdot)} \leq c.$$

By a similar argument to the estimate for $I_P$, we see that $J_P^1 \leq c$. For the converse estimate, it suffices to show that

$$\Big\| \Big| \frac{\Phi * f}{|P|^{\tau(\cdot)}} \Big|^{q(\cdot)} \chi_P \Big\|_{p(\cdot)/q(\cdot)} \leq c$$

for all $P \in \mathcal{Q}$ with $l(P) \geq 1$ and all $f \in \mathfrak{F}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}$ with $\|f\|^*_{\mathfrak{F}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}} \leq 1$. This claim can be reformulated as showing that $\Big\| \frac{\Phi * f}{|P|^{\tau(\cdot)}} \chi_P \Big\|_{p(\cdot)} \leq c$. Using the fact that there exist $\varrho_v \in \mathcal{S}(\mathbb{R}^n)$, $v = -\gamma, \cdots, 1$, such that $\Phi * f = \varrho_{-\gamma} * \Phi_{-\gamma} * f + \sum_{v=1-\gamma}^{1} \varrho_v * \varphi_v * f$, see [18, p. 130]. Applying Lemma 4 we obtain

$$\big\| \varrho_{-\gamma} * \Phi_{-\gamma} * f \big\|_{\widetilde{L_{\tau(\cdot)}^{p(\cdot)}}} \lesssim \|\Phi_{-\gamma} * f\|_{\widetilde{L_{\tau(\cdot)}^{p(\cdot)}}} \leq c,$$

and

$$\|\varrho_v * \varphi_v * f\|_{\widetilde{L_{\tau(\cdot)}^{p(\cdot)}}} \lesssim \|\varphi_v * f\|_{\widetilde{L_{\tau(\cdot)}^{p(\cdot)}}} \leq c, \quad v = 1 - \gamma, \cdots, 1,$$

by using (3) and the fact that $\|f\|^*_{\mathfrak{F}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}} \leq 1$. The proof is complete. ■

**Definition 2** *Let $p, q \in \mathcal{P}_0$, $\tau : \mathbb{R}^n \to \mathbb{R}^+$ and let $\alpha : \mathbb{R}^n \to \mathbb{R}$. Then for all complex valued sequences $\lambda = \{\lambda_{v,m} \in \mathbb{C}\}_{v \in \mathbb{N}_0, m \in \mathbb{Z}^n}$ we define*

$$\mathfrak{f}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)} := \Big\{ \lambda : \|\lambda\|_{\mathfrak{f}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}} < \infty \Big\},$$

*where*

$$\|\lambda\|_{\mathfrak{f}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}} := \sup_{P \in \mathcal{Q}} \Big\| \Big( \frac{\sum_{m \in \mathbb{Z}^n} 2^{v(\alpha(\cdot)+n/2)} \lambda_{v,m} \chi_{v,m}}{|P|^{\tau(\cdot)}} \chi_P \Big)_{v \geq v_P^+} \Big\|_{L^{p(\cdot)}(\ell^{q(\cdot)})}.$$

If we replace dyadic cubes $P$ by arbitrary balls $B_J$ of $\mathbb{R}^n$ with $J \in \mathbb{Z}$, we then obtain equivalent quasi-norms, where the supremum is taken over all $J \in \mathbb{Z}$ and all balls $B_J$ of $\mathbb{R}^n$. As in [15], we obtain the following two proprties.

**Lemma 13** *Let $\alpha, \tau \in C_{\mathrm{loc}}^{\log}$, $\tau^- \geq 0$, $p, q \in \mathcal{P}_0^{\log}$, $0 < q^+, p^+ < \infty$, $v \in \mathbb{N}_0, m \in \mathbb{Z}^n$, $x \in Q_{v,m}$ and $\lambda \in \mathfrak{f}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}$. Then there exists $c > 0$ independent of $v$ and $m$ such that*

$$|\lambda_{v,m}| \leq c \; 2^{-v(\alpha(x)+n/2)} |Q_{v,m}|^{\tau(x)} \, \|\lambda\|_{\mathfrak{f}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}} \, \big\| \chi_{v,m} \big\|_{p(\cdot)}^{-1}.$$

**Lemma 14** *Let $\alpha, \tau \in C_{\mathrm{loc}}^{\log}$, $\tau^- \geq 0$, $p, q \in \mathcal{P}_0^{\log}$ with $0 < q^+, p^+ < \infty$ and $\Psi$, $\psi \in \mathcal{S}(\mathbb{R}^n)$ satisfy, respectively, (4) and (5). Then for all $\lambda \in \mathfrak{f}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}$*

$$T_\psi \lambda := \sum_{m \in \mathbb{Z}^n} \lambda_{0,m} \Psi_m + \sum_{v=1}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{v,m} \psi_{v,m},$$

*converges in $\mathcal{S}'(\mathbb{R}^n)$; moreover, $T_\psi : \mathfrak{f}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)} \to \mathcal{S}'(\mathbb{R}^n)$ is continuous.*

For a sequence $\lambda = \{\lambda_{v,m} \in \mathbb{C}\}_{v\in\mathbb{N}_0, m\in\mathbb{Z}^n}$, $0 < r \leq \infty$ and a fixed $d > 0$, set

$$\lambda^*_{v,m,r,d} := \Big( \sum_{h\in\mathbb{Z}^n} \frac{|\lambda_{v,h}|^r}{(1+2^v|2^{-v}h - 2^{-v}m|)^d} \Big)^{1/r}$$

and $\lambda^*_{r,d} := \{\lambda^*_{v,m,r,d} \in \mathbb{C}\}_{v\in\mathbb{N}_0, m\in\mathbb{Z}^n}$.

**Lemma 15** *Let $\alpha, \tau \in C^{\log}_{\mathrm{loc}}$, $\tau^- > 0$, $p, q \in \mathcal{P}^{\log}_0$, $0 < q^+, p^+ < \infty$, $w > n + c_{\log}(1/q) + c_{\log}(\tau)$ and $\frac{\tau^+}{\tau^-} < \frac{\min(p^-,q^-)}{r}$. Let $a = r\max(c_{\log}(\alpha), \alpha^+ - \alpha^-)$. Then*

$$\left\|\lambda^*_{r,d}\right\|_{\mathfrak{f}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}} \approx \|\lambda\|_{\mathfrak{f}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}}$$

*where*

$$d > nr\tau^+ + n\tau^- + n + a + w.$$

The proof of this lemma is postponed to the Appendix. By this result and by the same arguments given in [15, Theorem 3.14] we obtain the following statement.

**Theorem 5** *Let $\alpha, \tau \in C^{\log}_{\mathrm{loc}}$, $\tau^- > 0$, $p, q \in \mathcal{P}^{\log}_0$ and $0 < q^+, p^+ < \infty$. Suppose that $\Phi$, $\Psi \in \mathcal{S}(\mathbb{R}^n)$ satisfying (4) and $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ satisfy (5) such that (9) holds. The operators $S_\varphi : \mathfrak{F}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)} \to \mathfrak{f}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$ and $T_\psi : \mathfrak{f}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)} \to \mathfrak{F}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$ are bounded. Furthermore, $T_\psi \circ S_\varphi$ is the identity on $\mathfrak{F}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$.*

From this theorem, we obtain the next important property of spaces $\mathfrak{F}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$.

**Corollary 1** *Let $\alpha, \tau \in C^{\log}_{\mathrm{loc}}$, $\tau^- > 0$, $p, q \in \mathcal{P}^{\log}_0$ and $0 < p^+, q^+ < \infty$, The definition of the spaces $\mathfrak{F}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$ is independent of the choices of $\Phi$ and $\varphi$.*

# 4 Embeddings

For the spaces $\mathfrak{F}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$ introduced above we want to show some embedding theorems. We say a quasi-Banach space $A_1$ is continuously embedded in another quasi-Banach space $A_2$, $A_1 \hookrightarrow A_2$, if $A_1 \subset A_2$ and there is a $c > 0$ such that $\|f\|_{A_2} \leq c\|f\|_{A_1}$ for all $f \in A_1$. We begin with the following elementary embeddings.

**Theorem 6** *Let $\alpha, \tau \in C^{\log}_{\mathrm{loc}}$, $\tau^- > 0$ and $p, q, q_0, q_1 \in \mathcal{P}^{\log}_0$ with $p^+, q^+, q_0^+, q_1^+ < \infty$.*
*(i) If $q_0 \leq q_1$, then*

$$\mathfrak{F}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q_0(\cdot)} \hookrightarrow \mathfrak{F}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q_1(\cdot)}.$$

*(ii) If $(\alpha_0 - \alpha_1)^- > 0$, then*

$$\mathfrak{F}^{\alpha_0(\cdot),\tau(\cdot)}_{p(\cdot),q_0(\cdot)} \hookrightarrow \mathfrak{F}^{\alpha_1(\cdot),\tau(\cdot)}_{p(\cdot),q_1(\cdot)}.$$

The proof can be obtained by using the properties of $L^{p(\cdot)}(\ell^{q(\cdot)})$ spaces. We next consider embeddings of Sobolev-type. It is well-known that

$$F^{\alpha_0,\tau}_{p_0,q} \hookrightarrow F^{\alpha_1,\tau}_{p_1,q}.$$

if $\alpha_0 - n/p_0 = \alpha_1 - n/p_1$, where $0 < p_0 < p_1 < \infty, 0 \leq \tau < \infty$ and $0 < q \leq \infty$ (see e.g. [48, Corollary 2.2]). In the following theorem we generalize these embeddings to variable exponent case.

**Theorem 7** *Let $\alpha_0, \alpha_1, \tau \in C^{\log}_{\mathrm{loc}}$, $\tau^- > 0$ and $p_0, p_1, q \in \mathcal{P}^{\log}_0$ with $0 < p_0^+, p_1^+, q^+ < \infty$. If $\alpha_0 > \alpha_1$ and $\alpha_0(x) - \frac{n}{p_0(x)} = \alpha_1(x) - \frac{n}{p_1(x)}$ with $0 < \left(\frac{p_0}{p_1}\right)^+ < 1$, then*

$$\mathfrak{F}^{\alpha_0(\cdot),\tau(\cdot)}_{p_0(\cdot),q(\cdot)} \hookrightarrow \mathfrak{F}^{\alpha_1(\cdot),\tau(\cdot)}_{p_1(\cdot),\infty}.$$

**Proof.** We use some idea of [31]. By homogeneity, it suffices to consider the case $\|f\|_{\mathfrak{F}^{\alpha_0(\cdot),\tau(\cdot)}_{p_0(\cdot),\infty}} = 1$. We will provet that

$$\Big\|\Big(\frac{2^{v\alpha_1(\cdot)}\varphi_v * f}{|P|^{\tau(\cdot)}}\chi_P\Big)_{v\geq v_P^+}\Big\|_{L^{p_1(\cdot)}(\ell^{q(\cdot)})} \lesssim 1$$

for any dyadic cube $P$. We will study two cases in particular.
*Case 1.* $|P| > 1$. Let $Q_v \subset P$ be a cube, with $\ell(Q_v) = 2^{-v}$ and $x \in Q_v \subset P$. By Lemma 2 we have for any $m > n$, $0 < r < \frac{\tau^- p_0^-}{\tau^+}$

$$|\varphi_v * f(x)| \leq c\left(\eta_{v,m} * |\varphi_v * f|^r(x)\right)^{1/r},$$

where $c > 0$ independent of $v$. We have

$$\begin{aligned}
&\eta_{v,m} * |\varphi_v * f|^r(x) \\
=\ & 2^{vn}\int_{\mathbb{R}^n} \frac{|\varphi_v * f(z)|^r}{(1 + 2^v|x-z|)^m}dz \\
=\ & \int_{3Q_v} \cdots dz + \sum_{k=(k_1,\ldots,k_n)\in\mathbb{Z}^n, \max_{i=1,\ldots,n}|k_i|\geq 2} \int_{Q_v + kl(Q_v)} \cdots dz.
\end{aligned}$$

Let $z \in Q_v + kl(Q_v)$ with $k \in \mathbb{Z}^n$ and $|k| > 4\sqrt{n}$. Then $|x - z| \geq |k|\, 2^{-v-1}$ and the second integral is boundd by

$$|k|^{-m}\, 2^{vn}\int_{Q_v + kl(Q_v)} |\varphi_v * f(z)|^r dz = |k|^{-m}\, M_{Q_v + kl(Q_v)}|\varphi_v * f|^r(x).$$

We put $\sigma = \frac{p_0}{p_1} \in ]0,1[$ and $s_0 = \alpha_1 - n/p_1 \in C^{\log}_{\mathrm{loc}}$. Since $\sigma\alpha_0 + (1-\sigma)\, s_0 = \alpha_1$, Lemma 10 gives for any $x \in \mathbb{R}^n$

$$\begin{aligned}
&\Big\|\Big(\frac{2^{\alpha_1(x)v}\varphi_v * f(x)}{|P|^{\tau(x)}}\chi_P(x)\Big)_{v\geq 0}\Big\|_{\ell^{q(x)}} \\
\leq\ & \Big\|\Big(\frac{2^{\alpha_0(x)v}\varphi_v * f(x)}{|P|^{\tau(x)}}\chi_P(x)\Big)_{v\geq 0}\Big\|^{\sigma(x)}_{\ell^\infty}\Big\|\Big(\frac{2^{s_0(x)v}\varphi_v * f(x)}{|P|^{\tau(x)}}\chi_P(x)\Big)_{v\geq 0}\Big\|^{1-\sigma(x)}_{\ell^\infty}.
\end{aligned}$$

This expression in $L^{p_1(\cdot)}$-norm is bounded. Indeed,

$$\begin{aligned}
&\Big\|\Big(\frac{2^{s_0(x)vr}\,|\varphi_v * f(x)|^r}{|P|^{\tau(x)r}}\chi_P(x)\Big)_{v\geq 0}\Big\|_{\ell^\infty} \\
\lesssim\ & \sum_{k\in\mathbb{Z}^n, |k|\leq 4\sqrt{n}} \Big\|\Big(\frac{2^{s_0(x)vr}g_{v,k,m,N}(x)}{|P|^{\tau(x)r}}\chi_P(x)\Big)_{v\geq 0}\Big\|_{\ell^\infty} + \\
& \sum_{k\in\mathbb{Z}^n, |k|>4\sqrt{n}} |k|^{n-m+n\tau^+r+\sigma^+}\, \Big\|\Big(\frac{2^{s_0(x)vr}|k|^{-n}g_{v,k,m,N}(x)}{|P|^{\tau(x)r}}\chi_P(x)\Big)_{v\geq 0}\Big\|_{\ell^\infty},
\end{aligned}$$

where $\varrho(\cdot)=n-\frac{ndr}{p_0(\cdot)\tau(\cdot)}$ and

$$g_{v,k,m,N}=\begin{cases} M_{3Q_v}|\varphi_v*f|^r & \text{if} \quad |k|=0 \\ M_{Q_v+kl(Q_v)}|\varphi_v*f|^r & \text{if} \quad 0<|k|\leq 4\sqrt{n} \\ |k|^{-n\tau r}M_{Q_v+kl(Q_v)}|k|^{-\varrho(\cdot)}|\varphi_v*f|^r & \text{if} \quad |k|>4\sqrt{n}. \end{cases}$$

We will prove that

$$\Big\|\Big(\frac{2^{s_0(x)vr}|k|^{-n}g_{v,k,m,N}(x)}{|P|^{\tau(x)r}}\chi_P(x)\Big)_{v\geq 0}\Big\|_{\ell^\infty}\lesssim 1 \tag{10}$$

for any $k\in\mathbb{Z}^n$ with $|k|>4\sqrt{n}$. We take $d>0$ such that $\tau^+<d<\frac{\tau^-p_0^-}{r}$. Observe that $Q_v+kl(Q_v)\subset Q\left(x,|k|2^{1-v}\right)$, by Theorem 1,

$$\begin{aligned} &\frac{|k|^{-ndr}}{|P|^{dr}}\left(M_{Q(x,|k|2^{1-v})}|k|^{-\varrho(\cdot)}2^{vs_0r}|\varphi_v*f|^r(x)\right)^{d/\tau(x)} \\ \leq\ & M_{Q(x,|k|2^{1-v})}\Big(\frac{2^{v\frac{s_0dr}{\tau}}|\varphi_v*f|^{rd/\tau}}{|k|^{ndr}\,|P|^{dr}}\Big)(x)+C. \end{aligned}$$

By Hölder's inequality this expression is bounded by

$$\Big\|\frac{2^{v\left(s_0+\frac{n}{p_0}\right)dr/\tau}}{|k|^{ndr}\,|P|^{dr}}|\varphi_v*f|^{rd/\tau}\chi_{Q(\cdot,|k|2^{1-v})}\Big\|_{p_0(\cdot)\tau(\cdot)/dr}+C.$$

Observe that $Q\left(x,|k|2^{1-v}\right)\subset Q\left(c_P,|k|2^{1-v_P}\right)$ for any $x\in Q_v$. The last norm is bounded if and only if

$$\int_{Q\left(c_P,|k|2^{1-v_P}\right)}\frac{2^{v\left(s_0(y)+\frac{n}{p_0(y)}\right)p_0(y)}|\varphi_v*f\left(y\right)|^{p_0(y)}}{\left(|k|^n\,|P|\right)^{p_0(y)\tau(y)}}dy\lesssim 1,$$

which is equivalent to

$$\Big\|\frac{2^{v\left(s_0(\cdot)+\frac{n}{p_0(\cdot)}\right)p_0(\cdot)}\,|\varphi_v*f|\,\chi_{Q\left(c_P,|k|2^{1-v_P}\right)}}{|Q\left(c_P,|k|2^{1-v_P}\right)|^{\tau(\cdot)}}\Big\|_{p_0(\cdot)}\lesssim 1,$$

due to the fact that $s_0(\cdot)+\frac{n}{p_0(\cdot)}=\alpha_0(\cdot)$ and $|Q\left(c_P,|k|2^{1-v_P}\right)|\geq 1$. Therefore, the sum $\sum_{k\in\mathbb{Z}^n,|k|>4\sqrt{n}}\cdots$ is bounded by taking $m$ large enough. Similar arguments with necessary modifications can be used to prove (10) for any $|k|\leq 4\sqrt{n}$. Therefore,

$$\begin{aligned} &\int_P\Big(\sum_{v=0}^{\infty}2^{v\alpha_1(x)q(x)}\frac{|\varphi_v*f(x)|^{q(x)}}{|P|^{\tau(x)q(x)}}\Big)^{p_1(x)/q(x)}dx \\ \leq\ & c\int_P\Big\|\Big(\frac{2^{\alpha_0(x)v}\varphi_v*f(x)}{|P|^{\tau(x)}}\Big)_{v\geq 0}\Big\|_{\ell^\infty}^{\sigma(x)p_1(x)}dx \\ =\ & c\int_P\Big\|\Big(\frac{2^{\alpha_0(x)v}\varphi_v*f(x)}{|P|^{\tau(x)}}\Big)_{v\geq 0}\Big\|_{\ell^\infty}^{p_0(x)}dx\leq c. \end{aligned}$$

*Case 2.* $|P| \leq 1$. Since $\tau$ is log-Hölder continuous, we have

$$|P|^{-\tau(x)} \leq c \ |P|^{-\tau(y)} \left(1 + 2^{v_P} |x-y|\right)^{c_{\log}(\tau)} \leq c \ |P|^{-\tau(y)} \left(1 + 2^{v} |x-y|\right)^{c_{\log}(\tau)}$$

for any $x, y \in \mathbb{R}^n$ and any $v \geq v_P$. Therefore,

$$\frac{1}{|P|^{\tau(\cdot)}} \eta_{v,m} * \left(|\varphi_v * f| \chi_Q\right) \lesssim \eta_{v,m-c_{\log}(\tau)} * \frac{|\varphi_v * f| \chi_Q}{|P|^{\tau(\cdot)}}$$

for any dyadic cube where $Q$. The arguments here are quite similar to those used in the case $|P| > 1$, where we did not need to use Theorem 1, which could be used only to move $|P|^{\tau(\cdot)}$ inside the convolution and hence the proof is complete. ■

Let $\alpha, \tau \in C^{\log}_{\mathrm{loc}}, \tau^- > 0, p, q \in \mathcal{P}^{\log}_0$ with $0 < p^+, q^+ < \infty$. From (8), we obtain

$$\mathfrak{F}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)} \hookrightarrow F^{\alpha(\cdot)+n\tau(\cdot)-\frac{n}{p(\cdot)}}_{p(\cdot),\infty} \hookrightarrow \mathcal{S}'(\mathbb{R}^n).$$

Similar arguments of [48, Proposition 2.3] can be used to prove that

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow \mathfrak{F}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}.$$

Therefore, we obtain the following result.

**Theorem 8** *Let $\alpha, \tau \in C^{\log}_{\mathrm{loc}}, \tau^- > 0$ and $p, q \in \mathcal{P}^{\log}_0$ with $0 < p^+, q^+ < \infty$. Then*

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow \mathfrak{F}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)} \hookrightarrow \mathcal{S}'(\mathbb{R}^n).$$

Now we establish some further embedding of the spaces $\mathfrak{F}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$.

**Theorem 9** *Let $\alpha, \tau \in C^{\log}_{\mathrm{loc}}, \tau^- > 0$ and $p, q \in \mathcal{P}^{\log}_0$ with $0 < p^+, q^+ < \infty$. If $(p_2 - p_1)^+ \leq 0$, then*

$$F^{\alpha(\cdot)+n\tau(\cdot)+\frac{n}{p_2(\cdot)}-\frac{n}{p_1(\cdot)}}_{p_2(\cdot),q(\cdot)} \hookrightarrow \mathfrak{F}^{\alpha(\cdot),\tau(\cdot)}_{p_1(\cdot),q(\cdot)}.$$

**Proof.** Using the Sobolev embeddings

$$F^{\alpha(\cdot)+n\tau(\cdot)+\frac{n}{p_2(\cdot)}-\frac{n}{p_1(\cdot)}}_{p_2(\cdot),q(\cdot)} \hookrightarrow F^{\alpha(\cdot)+n\tau(\cdot)}_{p_1(\cdot),q(\cdot)},$$

see [42] it is sufficient to prove that $F^{\alpha(\cdot)+n\tau(\cdot)}_{p_1(\cdot),q(\cdot)} \hookrightarrow \mathfrak{F}^{\alpha(\cdot),\tau(\cdot)}_{p_1(\cdot),q(\cdot)}$. We have

$$\sup_{P \in \mathcal{Q}, |P|>1} \Big\| \Big( \frac{2^{v\alpha(\cdot)} \varphi_v * f}{|P|^{\tau(\cdot)}} \chi_P \Big)_{v \geq v_P^+} \Big\|_{L^{p_1(\cdot)}(\ell^{q(\cdot)})} \leq \Big\| \big( 2^{v\alpha(\cdot)} \varphi_v * f \big)_{v \geq 0} \Big\|_{L^{p_1(\cdot)}(\ell^{q(\cdot)})}.$$

In view of the definition of $F^{\alpha(\cdot)}_{p_1(\cdot),q(\cdot)}$ spaces the last expression is bounded by $\|f\|_{F^{\alpha(\cdot)}_{p_1(\cdot),q(\cdot)}} \leq \|f\|_{F^{\alpha(\cdot)+n\tau(\cdot)}_{p_1(\cdot),q(\cdot)}}$. Now we have the estimates

$$\begin{aligned} & \sup_{P \in \mathcal{Q}, |P|\leq 1} \Big\| \Big( \frac{2^{v\alpha(\cdot)} \varphi_v * f}{|P|^{\tau(\cdot)}} \chi_P \Big)_{v \geq v_P^+} \Big\|_{L^{p_1(\cdot)}(\ell^{q(\cdot)})} \\ \leq \ & \sup_{P \in \mathcal{Q}, |P|\leq 1} \Big\| \Big( 2^{v(\alpha(\cdot)+n\tau(\cdot))+n\tau(\cdot)(v_P - v)} \varphi_v * f \Big)_{v \geq v_P} \Big\|_{L^{p_1(\cdot)}(\ell^{q(\cdot)})} \\ \leq \ & \sup_{P \in \mathcal{Q}, |P|\leq 1} \Big\| \big( 2^{v(\alpha(\cdot)+n\tau(\cdot))} \varphi_v * f \big)_{v \geq 0} \Big\|_{L^{p_1(\cdot)}(\ell^{q(\cdot)})} \leq \|f\|_{F^{\alpha(\cdot)+n\tau(\cdot)}_{p_1(\cdot),q(\cdot)}}, \end{aligned}$$

which completes the proof. ■

Let $p, u \in \mathcal{P}_0$ be such that $0 < p(\cdot) \leq u(\cdot) < \infty$. The variable Morrey space $M_{p(\cdot)}^{u(\cdot)}$ is defined to be the set of all $p(\cdot)$-locally Lebesgue-integrable functions $f$ on $\mathbb{R}^n$ such that

$$\|f\|_{M_{p(\cdot)}^{u(\cdot)}} = \sup_{P \in \mathcal{Q}} \Big\| \frac{f}{|P|^{\frac{1}{p(\cdot)} - \frac{1}{u(\cdot)}}} \chi_P \Big\|_{p(\cdot)} < \infty.$$

One recognizes immediately that if $p$ and $u$ are constants, then we obtain the usual Morrey space $M_p^u$.

Let $\alpha : \mathbb{R}^n \to \mathbb{R}$ and $p, q, u \in \mathcal{P}_0$ be such that $0 < p(\cdot) \leq u(\cdot) < \infty$. Let $\Phi$ and $\varphi$ satisfy (4) and (5), respectively and we put $\varphi_v = 2^{vn}\varphi(2^v \cdot)$. The Triebel-Lizorkin-Morrey space $\mathcal{E}_{u(\cdot),p(\cdot)}^{\alpha(\cdot),q(\cdot)}$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{\mathcal{E}_{u(\cdot),p(\cdot)}^{\alpha(\cdot),q(\cdot)}} := \Big\| \Big(2^{v\alpha(\cdot)} \varphi_v * f\Big)_{v \geq 0} \Big\|_{M_{p(\cdot)}^{u(\cdot)}(\ell^{q(\cdot)})} < \infty,$$

where $\varphi_0$ is replaced by $\Phi$.

**Theorem 10** *Let $\alpha \in C_{\mathrm{loc}}^{\log}$ and $p, q, u \in \mathcal{P}_0^{\log}$ with $0 < p^- < p(\cdot) \leq u(\cdot) < u^+ < \infty$ and $0 < q^- \leq q^+ < \infty$. Then*

$$\mathfrak{F}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\frac{1}{p(\cdot)} - \frac{1}{u(\cdot)}} = \mathcal{E}_{u(\cdot),p(\cdot)}^{\alpha(\cdot),q(\cdot)},$$

*with equivalent quasi-norms.*

**Proof.** Obviously we need to prove that $\|f\|_{\mathcal{E}_{u(\cdot),p(\cdot)}^{\alpha(\cdot),q(\cdot)}} \lesssim 1$ for any $f \in \mathfrak{F}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\frac{1}{p(\cdot)} - \frac{1}{u(\cdot)}}$ with $\|f\|_{\mathfrak{F}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\frac{1}{p(\cdot)} - \frac{1}{u(\cdot)}}} \leq 1$. We must show the inequality

$$\Big\| \Big( \sum_{v=0}^{v_P^+} \frac{\big|2^{v\alpha(\cdot)} \varphi_v * f\big|^{q(\cdot)}}{|P|^{(\frac{1}{p(\cdot)} - \frac{1}{u(\cdot)})q(\cdot)}} \chi_P \Big)^{1/q(\cdot)} \Big\|_{p(\cdot)} \lesssim 1$$

for any dyadic cube $P$. Applying the property (8), we obtain

$$2^{v\alpha(x)} |\varphi_v * f(x)| \lesssim 2^{\frac{vn}{u(x)}}$$

for any $x \in \mathbb{R}^n$, with the implicit constant independent of $k$ and $x$. Hence the last quasi-norm is bounded by

$$\Big\| 2^{\frac{nv_P^+}{p(\cdot)}} \Big( \sum_{v=0}^{v_P^+} 2^{(v - v_P^+)\frac{nq(\cdot)}{u(\cdot)}} \chi_P \Big)^{1/q(\cdot)} \Big\|_{L^{p(\cdot)}} \lesssim \Big\| 2^{\frac{nv_P^+}{p(\cdot)}} \chi_P \Big\|_{p(\cdot)} \lesssim 1.$$

This finishes the proof. ■

# 5 Atomic decomposition

The idea of atomic decompositions leads back to M. Frazier and B. Jawerth in their series of papers [17], [18], see also [38]. The main goal of this section is to prove an atomic decomposition result for $\mathfrak{F}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}$.
We now present a fundamental characterization of spaces under consideration.

**Theorem 11** *Let $\tau$, $\alpha \in C_{\mathrm{loc}}^{\log}, \tau^- > 0$ and $p, q \in \mathcal{P}_0^{\log}$ with $0 < p^- \leq p^+ < \infty$ and $0 < q^- \leq q^+ < \infty$. Let $m$ be as in Lemma 3, $a > \frac{\tau^+ m}{\tau^- p^-}$ and $\Phi$ and $\varphi$ satisfy* (4) *and* (5)*, respectively. Then*

$$\|f\|_{\mathfrak{F}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}}^{\blacktriangledown} := \sup_{P\in\mathcal{Q}} \Big\| \Big( \frac{\varphi_v^{*,a} 2^{v\alpha(\cdot)} f}{|P|^{\tau(\cdot)}} \chi_P \Big)_{v\geq v_P^+} \Big\|_{L^{p(\cdot)}(\ell^{q(\cdot)})}$$

*is an equivalent quasi-norm in $\mathfrak{F}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}$.*

**Proof.** It is easy to see that for any $f \in \mathcal{S}'(\mathbb{R}^n)$ with $\|f\|_{\mathfrak{F}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}}^{\blacktriangledown} < \infty$ and any $x \in \mathbb{R}^n$ we have

$$2^{v\alpha(x)} |\varphi_v * f(x)| \leq \varphi_v^{*,a} 2^{v\alpha(\cdot)} f(x).$$

This shows that $\|f\|_{\mathfrak{F}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}} \leq \|f\|_{\mathfrak{F}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}}^{\blacktriangledown}$. Let $t > 0$ be such that $a > \frac{m}{t} > \frac{m}{p^-}$. By Lemmas 2 and 1 the estimates

$$\begin{aligned} 2^{v\alpha(y)} |\varphi_v * f(y)| &\leq C_1 \, 2^{v\alpha(y)} \left(\eta_{v,w} * |\varphi_v * f|^t(y)\right)^{1/t} \\ &\leq C_2 \, \left(\eta_{v,w-c_{\log}(\alpha)} * (2^{v\alpha(\cdot)}|\varphi_v * f|)^t(y)\right)^{1/t} \end{aligned} \tag{11}$$

are true for any $y \in \mathbb{R}^n, v \in \mathbb{N}_0$ and any $w > 0$. Now divide both sides of (11) by $(1 + 2^v |x - y|)^a$, in the right-hand side we use the inequality

$$(1 + 2^v |x - y|)^{-a} \leq (1 + 2^v |x - z|)^{-a} (1 + 2^v |y - z|)^a, \quad x, y, z \in \mathbb{R}^n,$$

in the left-hand side take the supremum over $y \in \mathbb{R}^n$ and get for all $f \in \mathfrak{F}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}$, any $x \in P$ any $v \geq v_P^+$ and any $w > at + c_{\log}(\alpha)$

$$\left(\varphi_v^{*,a} 2^{v\alpha(\cdot)} f(x)\right)^t \leq C_2 \, \eta_{v,at} * (2^{v\alpha(\cdot)}|\varphi_v * f|)^t(x)$$

where $C_2 > 0$ is independent of $x, v$ and $f$. Lemma 3 gives that

$$\begin{aligned} \|f\|_{\mathfrak{F}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}}^{\blacktriangledown} &\lesssim C \sup_{P\in\mathcal{Q}} \Big\| \Big( \frac{\eta_{v,at} * (2^{v\alpha(\cdot)}|\varphi_v * f|)^t}{|P|^{\tau(\cdot)t}} \chi_P \Big)_{v\geq v_P^+} \Big\|_{L^{\frac{p(\cdot)}{t}}(\ell^{\frac{q(\cdot)}{t}})}^{1/t} \\ &\leq C \left\| \left(2^{v\alpha(\cdot)} \varphi_v * f\right)_v \right\|_{L^{p(\cdot)}(\ell^{q(\cdot)})} = C \|f\|_{\mathfrak{F}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}}. \end{aligned}$$

The proof is complete. ■
Atoms are the building blocks for the atomic decomposition.

**Definition 3** *Let $K \in \mathbb{N}_0, L+1 \in \mathbb{N}_0$ and let $\gamma > 1$. A $K$-times continuous differentiable function $a \in C^K(\mathbb{R}^n)$ is called $[K, L]$-atom centered at $Q_{v,m}$, $v \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$, if*

$$\text{supp } a \subseteq \gamma Q_{v,m} \tag{12}$$

$$|\partial^\beta a(x)| \leq 2^{v(|\beta|+1/2)}, \quad \text{for} \quad 0 \leq |\beta| \leq K, x \in \mathbb{R}^n \tag{13}$$

*and if*

$$\int_{\mathbb{R}^n} x^\beta a(x) dx = 0, \quad \text{for} \quad 0 \leq |\beta| \leq L \text{ and } v \geq 1. \tag{14}$$

If the atom $a$ located at $Q_{v,m}$, that means if it fulfills (12), then we will denote it by $a_{v,m}$. For $v = 0$ or $L = -1$ there are no moment conditions (14) required.

For proving the decomposition by atoms we need the following lemma, see Frazier & Jawerth [17, Lemma 3.3].

**Lemma 16** *Let $\Phi$ and $\varphi$ satisfy, respectively, (4) and (5) and let $\varrho_{v,m}$ be an $[K, L]$-atom. Then*

$$\left|\varphi_j * \varrho_{v,m}(x)\right| \leq c \; 2^{(v-j)K+vn/2} \left(1 + 2^v \left|x - x_{Q_{v,m}}\right|\right)^{-M}$$

*if $v \leq j$, and*

$$\left|\varphi_j * \varrho_{v,m}(x)\right| \leq c \; 2^{(j-v)(L+n+1)+vn/2} \left(1 + 2^j \left|x - x_{Q_{v,m}}\right|\right)^{-M}$$

*if $v \geq j$, where $M$ is sufficiently large, $\varphi_j = 2^{jn}\varphi(2^j \cdot)$ and $\varphi_0$ is replaced by $\Phi$.*

Now we come to the atomic decomposition theorem.

**Theorem 12** *Let $\alpha, \tau \in C^{\log}_{\text{loc}}, \tau^- > 0$ and $p, q \in \mathcal{P}^{\log}_0$ with $0 < p^- \leq p^+ < \infty$ and $0 < q^- \leq q^+ < \infty$ Let $K, L+1 \in \mathbb{N}_0$ such that*

$$K \geq ([\alpha^+ + n\tau^+] + 1)^+, \tag{15}$$

*and*

$$L \geq \max(-1, [n(\frac{\tau^+}{\tau^- \min(1, p^-, q^-)} - 1) - \alpha^-]). \tag{16}$$

*Then $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $\mathfrak{F}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$, if and only if it can be represented as*

$$f = \sum_{v=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{v,m} \varrho_{v,m}, \qquad \text{converging in } \mathcal{S}'(\mathbb{R}^n), \tag{17}$$

*where $\varrho_{v,m}$ are $[K, L]$-atoms and $\lambda = \{\lambda_{v,m} \in \mathbb{C} : v \in \mathbb{N}_0, m \in \mathbb{Z}^n\} \in \mathfrak{f}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$. Furthermore, $\inf \|\lambda\|_{\mathfrak{f}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}}$, where the infimum is taken over admissible representations (17), is an equivalent quasi-norm in $\mathfrak{F}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$.*

The convergence in $\mathcal{S}'(\mathbb{R}^n)$ can be obtained as a by-product of the proof using the same method as in [38, Corollary 13.9 ] and [15].

If $p$, $q$, $\tau$, and $\alpha$ are constants, then the restriction (15), and their counterparts, in the atomic decomposition theorem are $K \geq ([\alpha+n\tau]+1)^{+}$ and $L \geq \max(-1, [n(\frac{1}{\min(1,p,q)} - 1) - \alpha])$, which are essentially the restrictions from the works of [13, Theorem 3.12].
**Proof.** The proof follows the ideas in [17, Theorem 6].

*Step 1.* Assume that $f \in \mathfrak{F}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$ and let $\Phi$ and $\varphi$ satisfy, respectively (4) and (5). There exist functions $\Psi \in \mathcal{S}(\mathbb{R}^n)$ satisfying (4) and $\psi \in \mathcal{S}(\mathbb{R}^n)$ satisfying (5) such that for all $\xi \in \mathbb{R}^n$

$$f = \Psi * \widetilde{\Phi} * f + \sum_{v=1}^{\infty} \psi_v * \widetilde{\varphi}_v * f,$$

see Section 3. Using the definition of the cubes $Q_{v,m}$ we obtain

$$f(x) = \sum_{m\in\mathbb{Z}^n} \int_{Q_{0,m}} \widetilde{\Phi}(x-y)\Psi * f(y)dy + \sum_{v=1}^{\infty} 2^{vn} \sum_{m\in\mathbb{Z}^n} \int_{Q_{v,m}} \widetilde{\varphi}(2^v(x-y))\psi_v * f(y)dy,$$

with convergence in $\mathcal{S}'(\mathbb{R}^n)$. We define for every $v \in \mathbb{N}$ and all $m \in \mathbb{Z}^n$

$$\lambda_{v,m} = C_\theta \sup_{y\in Q_{v,m}} |\psi_v * f(y)| \tag{18}$$

where

$$C_\varphi = \max\{\sup_{|y|\leq 1} |D^\alpha \varphi(y)| : |\alpha| \leq K\}.$$

Define also

$$\varrho_{v,m}(x) = \begin{cases} \frac{1}{\lambda_{v,m}} 2^{vn} \int_{Q_{v,m}} \widetilde{\varphi}_v(2^v(x-y))\psi_v * f(y)dy & \text{if} \quad \lambda_{v,m} \neq 0 \\ 0 & \text{if} \quad \lambda_{v,m} = 0 \end{cases}. \tag{19}$$

Similarly we define for every $m \in \mathbb{Z}^n$ the numbers $\lambda_{0,m}$ and the functions $\varrho_{0,m}$ taking in (18) and (19) $v = 0$ and replacing $\psi_v$ and $\widetilde{\varphi}$ by $\Psi$ and $\widetilde{\Phi}$, respectively. Let us now check that such $\varrho_{v,m}$ are atoms in the sense of Definition 3. Note that the support and moment conditions are clear by (4) and (5), respectively. It thus remains to check (13) in Definition 3. We have

$$\begin{aligned} \left|D^\beta \varrho_{v,m}(x)\right| &\leq \frac{2^{v(n+|\beta|)}}{C_\varphi} \int_{Q_{v,m}} \left|(D^\beta\widetilde{\varphi})(2^v(x-y))\right| |\psi_v * f(y)|\, dy \Big(\sup_{y\in Q_{v,m}} |\psi_v * f(y)|\Big)^{-1} \\ &\leq \frac{2^{v(n+|\beta|)}}{C_\varphi} \int_{Q_{v,m}} \left|(D^\beta\widetilde{\varphi})(2^v(x-y))\right| dy \leq 2^{v(n+|\beta|)} \left|Q_{v,m}\right| \leq 2^{v|\beta|}. \end{aligned}$$

The modifications for the terms with $v = 0$ are obvious.

*Step 2.* Next we show that there is a constant $c > 0$ such that $\|\lambda\|_{\mathfrak{f}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}} \leq c \, \|f\|_{\mathfrak{F}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}}$. For that reason we exploit the equivalent quasi-norms given in Theorem 11 involving Peetre's maximal function. Let $v \in \mathbb{N}$. Taking into account that $|x-y| \leq c\ 2^{-v}$ for $x, y \in Q_{v,m}$ we obtain

$$2^{v(\alpha(x)-\alpha(y))} \leq \frac{c_{\log}(\alpha)v}{\log(e+1/|x-y|)} \leq \frac{c_{\log}(\alpha)v}{\log(e+2^v/c)} \leq c$$

if $v \geq [\log_2 c] + 2$. If $0 < v < [\log_2 c] + 2$, then $2^{v(\alpha(x)-\alpha(y))} \leq 2^{v(\alpha^+ - \alpha^-)} \leq c$. Therefore,

$$2^{v\alpha(x)} |\psi_v * f(y)| \leq c \; 2^{v\alpha(y)} |\psi_v * f(y)|$$

for any $x, y \in Q_{v,m}$ and any $v \in \mathbb{N}$. Hence,

$$\begin{aligned}
\sum_{m\in\mathbb{Z}^n} \lambda_{v,m} 2^{v\alpha(x)} \chi_{v,m}(x) &= C_\theta \sum_{m\in\mathbb{Z}^n} 2^{v\alpha(x)} \sup_{y\in Q_{v,m}} |\psi_v * f(y)| \chi_{v,m}(x) \\
&\leq c \sum_{m\in\mathbb{Z}^n} \sup_{|z|\leq c\, 2^{-v}} \frac{2^{v\alpha(x-z)} |\psi_v * f(x-z)|}{(1+2^v |z|)^a} (1+2^v |z|)^a \chi_{v,m}(x) \\
&\leq c\; \psi_v^{*,a} 2^{v\alpha(\cdot)} f(x) \sum_{m\in\mathbb{Z}^n} \chi_{v,m}(x) = c\; \psi_v^{*,a} 2^{v\alpha(\cdot)} f(x),
\end{aligned}$$

where we have used $\sum_{m\in\mathbb{Z}^n} \chi_{v,m}(x) = 1$. This estimate and its counterpart for $v = 0$ (which can be obtained by a similar calculation) give

$$\|\lambda\|_{\mathfrak{f}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}} \leq c \left\| \left(\psi_v^{*,a} 2^{v\alpha(\cdot)} f\right)_v \right\|_{L^{\tau(\cdot)}_{p(\cdot)}(\ell^{q(\cdot)})} \leq c \|f\|_{\mathfrak{F}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}},$$

by Theorem 11.

*Step* 3. Assume that $f$ can be represented by (17), with $K$ and $L$ satisfying (15) and (16), respectively. We will show that $f \in \mathfrak{F}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$ and that for some $c > 0$, $\|f\|_{\mathfrak{F}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}} \leq c \|\lambda\|_{\mathfrak{f}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}}$. The arguments are very similar to those in [15]. For the convenience of the reader, we give some details. We write

$$f = \sum_{v=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{v,m} \varrho_{v,m} = \sum_{v=0}^{j} \cdots + \sum_{v=j+1}^{\infty} \cdots.$$

From Lemmas 16 and 6, we have for any $M$ sufficiently large and any $v \leq j$

$$\begin{aligned}
&\sum_{m\in\mathbb{Z}^n} 2^{j\alpha(x)} |\lambda_{v,m}| \left|\varphi_j * \varrho_{v,m}(x)\right| \\
\lesssim\;& 2^{(v-j)(K-\alpha^+)} \sum_{m\in\mathbb{Z}^n} 2^{v(\alpha(x)-n/2)} |\lambda_{v,m}| \eta_{v,M}(x - x_{Q_{v,m}}) \\
\lesssim\;& 2^{(v-j)(K-\alpha^+)} \sum_{m\in\mathbb{Z}^n} 2^{v(\alpha(x)+n/2)} |\lambda_{v,m}| \eta_{v,M} * \chi_{v,m}(x).
\end{aligned}$$

Lemma 1 gives $2^{v\alpha(\cdot)} \eta_{v,M} * \chi_{v,m} \lesssim \eta_{v,T} * 2^{v\alpha(\cdot)} \chi_{v,m}$, with $T = M - c_{\log}(\alpha)$ and since $K > \alpha^+ + \tau^+$ we apply Lemma 9 to obtain

$$\begin{aligned}
&\left\| \left( \sum_{v=0}^{j} 2^{(v-j)(K-\alpha^+)} \eta_{v,T} * \left[ 2^{v(\alpha(\cdot)+n/2)} \sum_{m\in\mathbb{Z}^n} |\lambda_{v,m}| \chi_{v,m} \right] \right)_j \right\|_{L^{\tau(\cdot)}_{p(\cdot)}(\ell^{q(\cdot)})} \\
\lesssim\;& \left\| \left( \eta_{v,T} * \left[ 2^{v(\alpha(\cdot)+n/2)} \sum_{m\in\mathbb{Z}^n} |\lambda_{v,m}| \chi_{v,m} \right] \right)_v \right\|_{L^{\tau(\cdot)}_{p(\cdot)}(\ell^{q(\cdot)})}.
\end{aligned}$$

The right-hand side can be rewritten us

$$
\begin{aligned}
&\sup_{P\in\mathcal{Q}} \Big\| \Big( \frac{\Big(\eta_{v,T} * \Big[2^{v(\alpha(\cdot)+n/2)} \sum_{m\in\mathbb{Z}^n} |\lambda_{v,m}|\, \chi_{v,m}\Big]\Big)^r}{|P|^{r\tau(\cdot)}} \chi_P \Big)_{v\geq v_P^+} \Big\|^{1/r}_{L^{p(\cdot)/r}(\ell^{q(\cdot)/r})} \\
\lesssim\ &\sup_{P\in\mathcal{Q}} \Big\| \Big( \frac{\eta_{v,Tr} * \Big[2^{v(\alpha(\cdot)+n/2)r} \sum_{m\in\mathbb{Z}^n} |\lambda_{v,m}|^r\, \chi_{v,m}\Big]}{|P|^{r\tau(\cdot)}} \chi_P \Big)_{v\geq v_P^+} \Big\|^{1/r}_{L^{p(\cdot)/r}(\ell^{q(\cdot)/r})},
\end{aligned}
$$

by Lemma 7, since $\eta_{v,T} \approx \eta_{v,T} * \eta_{v,T}$ and $0 < r < \frac{\tau^-}{\tau^+} \min(1, p^-, q^-)$. The application of Lemma 3 and the fact that

$$
\Big\| (g_v)_{v\geq v_P^+} \Big\|^{1/r}_{L^{p(\cdot)/r}(\ell^{q(\cdot)/r})} = \Big\| \big(|g_v|^{1/r}\big)_{v\geq v_P^+} \Big\|_{L^{p(\cdot)}(\ell^{q(\cdot)})}
$$

give that the last expression is bounded by $\|\lambda\|_{\mathfrak{f}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}}$. Now from Lemma 16, we have for any $M$ sufficiently large and $v \geq j$

$$
\begin{aligned}
&\sum_{m\in\mathbb{Z}^n} 2^{j\alpha(x)} |\lambda_{v,m}| \left|\varphi_j * \varrho_{v,m}(x)\right| \\
\lesssim\ & 2^{(j-v)(L+1+n/2)} \sum_{m\in\mathbb{Z}^n} 2^{j(\alpha(x)-n/2)} |\lambda_{v,m}|\, \eta_{j,M}(x - x_{Q_{v,m}}) \\
\lesssim\ & 2^{(j-v)(L+1+n/2)} \sum_{m\in\mathbb{Z}^n} 2^{j(\alpha(x)-n/2)} |\lambda_{v,m}|\, \eta_{j,M} * \eta_{v,M}(x - x_{Q_{v,m}}),
\end{aligned}
$$

where the last inequality follows by Lemma 5, since $\eta_{j,M} = \eta_{\min(v,j),M}$. Again by Lemma 6, we have

$$
\eta_{j,M} * \eta_{v,M}(x - x_{Q_{v,m}}) \lesssim 2^{vn} \eta_{j,M} * \eta_{v,M} * \chi_{v,m}(x).
$$

Therefore, $\sum_{m\in\mathbb{Z}^n} 2^{j\alpha(x)} |\lambda_{v,m}| \left|\varphi_j * \varrho_{v,m}(x)\right|$ is bounded by

$$
\begin{aligned}
& c\ 2^{(j-v)(L+1-n/2)} \sum_{m\in\mathbb{Z}^n} 2^{j(\alpha(x)+n/2)} |\lambda_{v,m}|\, \eta_{j,M} * \eta_{v,M} * \chi_{v,m}(x) \\
\lesssim\ & 2^{(j-v)(L+1-\alpha^-)} \eta_{j,T} * \eta_{v,T} * \Big[2^{v(\alpha(\cdot)+n/2)} \sum_{m\in\mathbb{Z}^n} |\lambda_{v,m}|\, \chi_{v,m}\Big](x),
\end{aligned}
$$

by Lemma 1, with $T = M - c_{\log}(\alpha)$. Let $0 < r < \frac{\tau^-}{\tau^+} \min(1, p^-, q^-)$ be a real number such that $L > n/r - 1 - \alpha^- - n$. We have

$$
\begin{aligned}
&\Big( \sum_{v=j}^{\infty} 2^{(j-v)(L+1-\alpha^-)} \eta_{j,T} * \eta_{v,T} * \Big[2^{v(\alpha(\cdot)+n/2)} \sum_{m\in\mathbb{Z}^n} |\lambda_{v,m}|\, \chi_{v,m}\Big]\Big)^r \\
\leq\ & \sum_{v=j}^{\infty} 2^{(j-v)(L-n/r+1-\alpha^-+n)r} \eta_{j,Tr} * \eta_{v,Tr} * \Big[2^{v(\alpha(\cdot)+n/2)r} \sum_{m\in\mathbb{Z}^n} |\lambda_{v,m}|^r\, \chi_{v,m}\Big],
\end{aligned}
$$

where we have used Lemma 7. The application of Lemma 3 gives that

$$
\Big\| \Big( \sum_{v=j}^{\infty} 2^{(j-v)(L+1-\alpha^-)} \eta_{j,T} * \eta_{v,T} * \Big[2^{v(\alpha(\cdot)+n/2)} \sum_{m\in\mathbb{Z}^n} |\lambda_{v,m}|\, \chi_{v,m}\Big]\Big)_j \Big\|_{L^{\tau(\cdot)}_{p(\cdot)}(\ell^{q(\cdot)})}
$$

is bounded by

$$c \sup_{P\in\mathcal{Q}} \Big\| \Big( \frac{\sum_{v=j}^{\infty} 2^{(j-v)Hr} \eta_{v,Tr} * \Big[ 2^{v(\alpha(\cdot)+n/2)r} \sum\limits_{m\in\mathbb{Z}^n} |\lambda_{vm}|^r \chi_{v,m} \Big]}{|P|^{r\tau(\cdot)}} \chi_P \Big)_{j\geq j_P^+} \Big\|_{L^{p(\cdot)/r}(\ell^{q(\cdot)/r})}^{1/r},$$

where $H := L - n/r + n + 1 - \alpha^-$. Observing that $H > 0$, an application of Lemma 9, yields that the last expression is bounded by

$$c \sup_{P\in\mathcal{Q}} \Big\| \Big( \frac{\eta_{v,Tr} * \Big[ 2^{v(\alpha(\cdot)+n/2)r} \sum\limits_{m\in\mathbb{Z}^n} |\lambda_{v,m}|^r \chi_{v,m} \Big]}{|P|^{r\tau(\cdot)}} \chi_P \Big)_{v\geq v_P^+} \Big\|_{L^{p(\cdot)/r}(\ell^{q(\cdot)/r})}^{1/r} \lesssim \|\lambda\|_{\mathfrak{f}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}},$$

where we used again Lemma 3 and hence the proof is complete. ■

# 6 Appendix

Here we present more technical proofs of the Lemmas.

*Proof of Lemma* 3. By the scaling argument, we see that it suffices to consider when $\|(f_v)_v\|_{L_{p(\cdot)}^{\tau(\cdot)}(\ell^{q(\cdot)})} \leq 1$ and show that for any dyadic cube $P$

$$\Big\| \Big( \sum_{v=v_P^+}^{\infty} \Big| \frac{\eta_{v,m} * |f_v|}{|P|^{\tau(\cdot)}} \Big|^{q(\cdot)} \Big)^{1/q(\cdot)} \chi_P \Big\|_{p(\cdot)} \lesssim 1. \tag{20}$$

*Case 1.* $|P| > 1$. Let $Q_v \subset P$ be a cube, with $\ell(Q_v) = 2^{-v}$ and $x \in Q_v \subset P$. We have

$$\begin{aligned} & \eta_{v,m} * |f_v|(x) \\ = \;& 2^{vn} \int_{\mathbb{R}^n} \frac{|f_v(z)|}{(1 + 2^v |x - z|)^m} dz \\ = \;& \int_{3Q_v} \cdots dz + \sum_{k=(k_1,\ldots,k_n)\in\mathbb{Z}^n, \max_{i=1,\ldots,n} |k_i| \geq 2} \int_{Q_v + kl(Q_v)} \cdots dz \\ = \;& J_v^1(f_v \chi_{3Q_v})(x) + \sum_{k=(k_1,\ldots,k_n)\in\mathbb{Z}^n, \max_{i=1,\ldots,n} |k_i| \geq 2} J_{v,k}^2(f_v \chi_{Q_v + kl(Q_v)})(x). \end{aligned} \tag{21}$$

Let $z \in Q_v + kl(Q_v)$ with $k \in \mathbb{Z}^n$ and $|k| > 4\sqrt{n}$. Then $|x - z| \geq |k| 2^{-v-1}$ and

$$J_{v,k}^2(f_v \chi_{Q_v + kl(Q_v)})(x) \lesssim |k|^{-m} M_{Q_v + kl(Q_v)}(f_v)(x).$$

Let $d > 0$ be such that $\tau^+ < d \leq \tau^- \min(p^-, q^-)$. Therefore, the left-hand side of (20) is bounded by

$$\begin{aligned} & \sum_{k\in\mathbb{Z}^n, |k|\leq 4\sqrt{n}} \Big\| \Big( \sum_{v=0}^{\infty} \Big| \frac{g_{v,k}}{|P|^{\tau(\cdot)}} \Big|^{q(\cdot)} \Big)^{1/q(\cdot)} \chi_P \Big\|_{p(\cdot)} \\ & + \sum_{k\in\mathbb{Z}^n, |k|> 4\sqrt{n}} |k|^{w-m+n\left(1+\frac{1}{t^-}\right)\tau^+} \log|k| \Big\| \Big( \sum_{v=0}^{\infty} \Big| \frac{g_{v,k}}{|P|^{\tau(\cdot)}} \Big|^{q(\cdot)} \Big)^{1/q(\cdot)} \chi_P \Big\|_{p(\cdot)}, \end{aligned} \tag{22}$$

where

$$g_{v,k}=\begin{cases} M_{3Q_v}\left(f_v\right) & \text{if} \quad k=0 \\ M_{Q_v+kl(Q_v)}\left(f_v\right) & \text{if} \quad 0<|k|\leq 4\sqrt{n} \\ M_{Q_v+kl(Q_v)}\left(|k|^{-n(1+1/t(\cdot))\tau(\cdot)}f_v\right) & \text{if} \quad |k|>4\sqrt{n}. \end{cases}$$

with $\frac{1}{d}=\frac{1}{p(\cdot)\tau(\cdot)}+\frac{1}{t(\cdot)}$. By similarity we only estimate the second norm in (22). This term is bounded if and only if

$$\Big\|\Big(\sum_{v=0}^{\infty}\Big|\frac{(b_k g_{v,k})^{d/\tau(\cdot)}}{|P|^d}\Big|^{q(\cdot)\tau(\cdot)/d}\Big)^{d/\tau(\cdot)q(\cdot)}\chi_P\Big\|_{p(\cdot)\tau(\cdot)/d}\lesssim 1, \tag{23}$$

where $b_k=\left(\frac{1}{|k|^w\log|k|}\right)^{\tau(\cdot)/d}$. Observe that $Q_v+kl(Q_v)\subset Q(x,|k|\,2^{1-v})$. By Hölder's inequality,

$$\begin{aligned} &\left|Q(x,|k|\,2^{1-v})\right| M_{Q(x,|k|2^{1-v})}\left(|k|^{-n(1+1/t(\cdot))d}\,|f_v|^{d/\tau(\cdot)}\right)(x) \\ \lesssim\ &\Big\|\frac{|f_v|^{1/\tau(\cdot)}}{|Q(x,|k|\,2^{1-v})|}\chi_{Q(x,|k|2^{1-v})}\Big\|_{p(\cdot)\tau(\cdot)}^{d}\Big\||Q(x,|k|\,2^{1-v})|\,|k|^{-n(1+1/t(\cdot))}\,\chi_{Q(x,|k|2^{1-v})}\Big\|_{t(\cdot)}^{d} \end{aligned}$$

for any $x\in Q(x,|k|\,2^{1-v})$. The second norm is bounded. The first norm is bounded if and only if

$$\Big\|\frac{f_v}{|Q(x,|k|\,2^{1-v})|^{\tau(\cdot)}}\chi_{Q(x,|k|2^{1-v})}\Big\|_{p(\cdot)}\lesssim 1,$$

which follows since $\|(f_v)_v\|_{L^{\tau(\cdot)}_{p(\cdot)}(\ell^{q(\cdot)})}\leq 1$. Therefore we can apply Lemma 1,

$$\begin{aligned} &\left(M_{Q(x,|k|2^{1-v})}\left(|k|^{-n(1+1/t(\cdot))\tau(\cdot)}\,f_v\right)(x)\right)^{d/\tau(x)} \\ \leq\ & M_{Q(x,|k|2^{1-v})}\Big(\Big||k|^{-n(1+1/t(\cdot))\tau(\cdot)}\,f_v\Big|^{d/\tau(\cdot)}\Big)(x)+\min\left(1,|k|^{ns}\,2^{n(1-v)s}\right)\omega(x) \end{aligned}$$

for any $s>0$ large enough, where

$$\begin{aligned} \omega(x) &= (e+|x|)^{-s}+M_{Q(x,|k|2^{1-v})}\left((e+|\cdot|)^{-s}\right)(x) \\ &\leq (e+|x|)^{-s}+\mathcal{M}\left((e+|\cdot|)^{-s}\right)(x)=h\left(x\right). \end{aligned}$$

Therefore $(g_{v,k})^{d/\tau(\cdot)}$ can be estimated by

$$c\ M_{Q(\cdot,|k|2^{1-v})}\left(|k|^{-nd}\,|f_v|^{d/\tau(\cdot)}\right)+\sigma_{v,k}\ h,$$

where

$$\sigma_{v,k}=\begin{cases} 1 & \text{if} \quad 2^v\leq 2\,|k| \\ |k|^{ns}\,2^{-vns} & \text{if} \quad 2^v>2\,|k|\,. \end{cases}$$

Therefore, the quantity $\left(\cdots\right)^{d/\tau(\cdot)q(\cdot)}$ of the term (23) is bounded by

$$\begin{aligned} &|k|^{-w}\Big(\sum_{v=0}^{\infty}\Big|M_{Q(\cdot,|k|2^{-v+1})}\Big(\frac{|k|^{-nd}\,|f_v|^{d/\tau(\cdot)}}{|P|^d}\chi_{Q(\cdot,|k|2^{-v+1})}\Big)\Big|^{q(\cdot)\tau(\cdot)/d}\Big)^{d/\tau(\cdot)q(\cdot)} \\ &+\frac{1}{\log|k|}\Big(\sum_{v=0}^{\infty}\left(\sigma_{v,k}\,|h|\right)^{q(\cdot)\tau(\cdot)/d}\Big)^{d/\tau(\cdot)q(\cdot)}. \end{aligned} \tag{24}$$

The first term is bounded by

$$\Big(\sum_{v=0}^{\infty}\Big|\eta_{v,w}*\Big(\frac{|k|^{-nd}|f_v|^{d/\tau(\cdot)}}{|P|^d}\chi_{Q(\cdot,|k|2^{1-v})}\Big)\Big|^{q(\cdot)\tau(\cdot)/d}\Big)^{d/\tau(\cdot)q(\cdot)}.$$

where $w > n + c_{\log}(1/q) + c_{\log}(\tau)$. Applying Theorem 2, the $L^{p(\cdot)\tau(\cdot)/d}$-norm of this expression is bounded by

$$\Big\|\Big(\sum_{v=0}^{\infty}\Big|\frac{|k|^{-n\tau(\cdot)}f_v}{|P|^{\tau(\cdot)}}\Big|^{q(\cdot)}\chi_{Q(\cdot,|k|2^{1-v})}\Big)^{d/\tau(\cdot)q(\cdot)}\Big\|_{p(\cdot)\tau(\cdot)/d}.$$

Observe that $Q(\cdot,|k|\,2^{-v+1}) \subset Q(c_P,|k|\,2^{-v_P+1})$ and the measure of the last cube is greater that 1, the last norm is bounded by 1. The summation (24) can be estimated by

$$\begin{aligned}&c\,|h|^{q(\cdot)\tau(\cdot)/d}\Big(c+\sum_{v\geq 1,2^v\leq 2|k|}\frac{1}{\log|k|}+\sum_{2^v>2|k|}\Big(\frac{2^v}{|k|}\Big)^{-nsq(\cdot)\tau(\cdot)/d}\Big)\\ \lesssim\ &|h|^{q(\cdot)\tau(\cdot)/d}.\end{aligned}$$

This expression, with power $d/\tau(\cdot)q(\cdot)$, in the $L^{p(\cdot)\tau(\cdot)/d}$-norm is bounded by 1. Therefore, the second sum in (22) is bounded by taking $m$ large enough such that $m > n\tau^+ + 2n + w$, with $w > n + c_{\log}(1/q) + c_{\log}(\tau)$.

*Case 2.* $|P| \leq 1$. As before,

$$\eta_{v,m}*|f_v|(x) \lesssim J_v^1(f_v\chi_{3P})(x) + \sum_{k=(k_1,\dots,k_n)\in\mathbb{Z}^n,\max_{i=1,\dots,n}|k_i|\geq 2} J_{v,k}^2(f_v\chi_{P+kl(P)})(x).$$

We see that

$$J_v^1(f_v\chi_{3P})(x) \lesssim \eta_{v,m}*(|f_v|\,\chi_{3P})\,(x).$$

Since $\tau$ is log-Hölder continuous,

$$|P|^{-\tau(x)} \leq c\,|P|^{-\tau(y)}\,(1+2^{v_P}\,|x-y|)^{c_{\log}(\tau)} \leq c\,|P|^{-\tau(y)}\,(1+2^{v}\,|x-y|)^{c_{\log}(\tau)}$$

for any $x \in P$ and any $y \in 3P$. Hence

$$|P|^{-\tau(x)}\,J_v^1(f_v\chi_{3P})(x) \lesssim \eta_{v,m-c_{\log}(\tau)}*\Big(|P|^{-\tau(\cdot)}\,|f_v|\,\chi_{3P}\Big)\,(x).$$

Also, we have

$$|P|^{-\tau(x)}\,J_{v,k}^2(f_v\chi_{P+kl(P)})(x) \lesssim \eta_{v,m-c_{\log}(\tau)}*\Big(|P|^{-\tau(\cdot)}\,|f_v|\,\chi_{P+kl(P)}\Big)\,(x)$$

for $x \in P$ and $z \in P+kl(P)$ with $k \in \mathbb{Z}^n$ and $|k| > 4\sqrt{n}$. Our estimate follows by Lemma 1. The proof is complete.

*Proof of Lemma* 15. Obviously, $\|\lambda\|_{\mathfrak{f}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}} \leq \big\|\lambda_{r,d}^*\big\|_{\mathfrak{f}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}}$. Let us prove the converse inequality. By the scaling argument, it suffices to consider the case $\|\lambda\|_{\mathfrak{f}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}} \leq 1$ and

show that the modular of a the sequence on the left-hand side is bounded. It suffices to prove that

$$\Big\|\Big(\sum_{v=v_P^+}^{\infty}\Big|\frac{\sum_{m\in\mathbb{Z}^n}2^{v(\alpha(\cdot)+n/2)}\lambda^*_{v,m,r,d}\chi_{v,m}}{|P|^{\tau(\cdot)}}\Big|^{q(\cdot)}\Big)^{1/q(\cdot)}\chi_P\Big\|_{p(\cdot)}\lesssim 1 \tag{25}$$

for any dyadic cube $P\in\mathcal{Q}$. For each $k\in\mathbb{N}_0$ we define $\Omega_k:=\{h\in\mathbb{Z}^n : 2^{k-1}<2^v\left|2^{-v}h-2^{-v}m\right|\leq 2^k\}$ and $\Omega_0:=\{h\in\mathbb{Z}^n : 2^v\left|2^{-v}h-2^{-v}m\right|\leq 1\}$. Then for any $x\in Q_{v,m}\cap P$, $\sum_{h\in\mathbb{Z}^n}\frac{2^{vr\alpha(x)}|\lambda_{v,h}|^r}{(1+2^v|2^{-v}h-2^{-v}m|)^d}$ can be rewritten as

$$\begin{aligned}
&\sum_{k=0}^{\infty}\sum_{h\in\Omega_k}\frac{2^{vr\alpha(x)}\left|\lambda_{v,h}\right|^r}{\left(1+2^v\left|2^{-v}h-2^{-v}m\right|\right)^d}\\
\lesssim\ &\sum_{k=0}^{\infty}2^{-dk}\sum_{h\in\Omega_k}2^{vr\alpha(x)}\left|\lambda_{v,h}\right|^r\\
=\ &\sum_{k=0}^{\infty}2^{(n-d)k+(v-k)n+vr\alpha(x)}\int\limits_{\cup_{z\in\Omega_k}Q_{v,z}}\sum_{h\in\Omega_k}\left|\lambda_{v,h}\right|^r\chi_{v,h}(y)dy.
\end{aligned}\tag{26}$$

Let $x\in Q_{v,m}\cap P$ and $y\in\cup_{z\in\Omega_k}Q_{v,z}$, then $y\in Q_{v,z}$ for some $z\in\Omega_k$ and $2^{k-1}<2^v\left|2^{-v}z-2^{-v}m\right|\leq 2^k$. From this it follows that $y$ is located in some cube $Q(x,2^{k-v+3})$. In addition, from the fact that

$$|y_i-(c_P)_i|\leq|y_i-x_i|+|x_i-(c_P)_i|\leq 2^{k-v+2}+2^{-v_P-1}<2^{k-v_P+3},\quad i=1,...,n,$$

we have $y$ is located in some cube $Q(c_P,2^{k-v_P+4})$. Since $\alpha$ is log-Hölder continuous we can prove that

$$2^{v(\alpha(x)-\alpha(y))}\lesssim\begin{cases}2^{c_{\log}(\alpha)k} & \text{if}\quad k<\max(0,v-h_n)\\ 2^{(\alpha^+-\alpha^-)k} & \text{if}\quad k\geq\max(0,v-h_n),\end{cases}$$

where $c>0$ not depending on $v$ and $k$. Therefore, (26) does not exceed

$$\begin{aligned}
&c\sum_{k=0}^{\infty}2^{(n-d+a)k+(v-k)n}\int\limits_{Q(x,2^{k-v+3})}2^{v\alpha(y)r}\sum_{h\in\Omega_k}\left|\lambda_{v,h}\right|^r\chi_{v,h}(y)\chi_{Q(c_P,2^{k-v_P+4})}(y)dy\\
\lesssim\ &\sum_{k=0}^{\infty}2^{(n-d+a)k}M_{Q(x,2^{k-v+3})}\Big(\sum_{h\in\Omega_k}2^{v\alpha(\cdot)r}\left|\lambda_{v,h}\right|^r\chi_{v,h}\chi_{Q(c_P,2^{k-v_P+4})}\Big)(x),
\end{aligned}$$

where $a=\max\left(c_{\log}(\alpha),\alpha^+-\alpha^-\right)$. To prove (25) we can distinguish two cases:

*Case 1.* $|P|>1$. The left-hand side of (25) is bounded by 1 if and only if

$$c\sum_{k=0}^{\infty}2^{\varrho k}\Big\|\Big(\sum_{v=0}^{\infty}\Big(\frac{M_{Q(\cdot,2^{k-v+3})}(2^{-kn(r+1/t(\cdot))\tau(\cdot)}g_{v,k,v_P})}{|P|^{r\tau(\cdot)}}\Big)^{q(\cdot)/r}\Big)^{r/q(\cdot)}\chi_P\Big\|_{p(\cdot)/r}\lesssim 1, \tag{27}$$

with

$$g_{v,k,v_P}=\sum_{h\in\Omega_k}2^{v(\alpha(\cdot)+n/2)r}\left|\lambda_{v,h}\right|^r\chi_{v,h}\chi_{Q(c_P,2^{k-v_P+4})}$$

and

$$\varrho = n - d + a + nr\tau^{+} + n\frac{\tau^{+}}{t^{-}}$$

Let us prove that

$$\Big\| \Big( \sum_{v=0}^{\infty} \Big( \frac{\omega_k M_{Q(\cdot,2^{k-v+3})}(2^{-kn(r+1/t(\cdot))\tau(\cdot)} g_{v,k,v_P})}{|P|^{r\tau(\cdot)}} \Big)^{q(\cdot)/r} \Big)^{r/q(\cdot)} \chi_P \Big\|_{p(\cdot)/r} \lesssim 1$$

for any $k, v \in \mathbb{N}_0$ and any $P \in \mathcal{Q}$, where

$$\omega_k = \frac{1}{2^{(w - \frac{n\sigma}{t^{+}})k} + 2^{kns}}.$$

This is equivalent to

$$\Big\| \Big( \sum_{v=0}^{\infty} \Big( \frac{\big(\omega_k M_{Q(\cdot,2^{k-v+3})}(2^{-kn(r+1/t(\cdot))\tau(\cdot)} g_{v,k,v_P})\big)^{\frac{\sigma}{\tau(\cdot)}}}{|P|^{r\sigma}} \Big)^{\frac{q(\cdot)\tau(\cdot)}{r\sigma}} \Big)^{\frac{\sigma r}{q(\cdot)\tau(\cdot)}} \Big\|_{\frac{p(\cdot)\tau(\cdot)}{r\sigma}} \lesssim 1, \qquad (28)$$

where $\sigma > 0$ such that $\tau^{+} < \sigma < \frac{\tau^{-} \min\left(p^{-}, q^{-}\right)}{r}$. By Hölder's inequality,

$$\begin{aligned} & |Q(x, 2^{k-v+3})| M_{Q(x,2^{k-v+3})} \left( 2^{-kn(r+1/t(\cdot))\sigma} \left| g_{v,k,v_P} \right|^{\sigma/\tau(\cdot)} \right)(x) \\ \lesssim \;& \Big\| \frac{\left| g_{v,k,v_P} \right|^{1/\tau(\cdot)}}{|Q(x, 2^{k-v+3})|^{r}} \chi_{Q(x,2^{k-v+3})} \Big\|_{p(\cdot)\tau(\cdot)/r}^{\sigma} \Big\| |Q(x, 2^{k-v+3})|^{r} 2^{-kn(r+1/t(\cdot))} \chi_{Q(x,2^{k-v+3})} \Big\|_{t(\cdot)}^{\sigma} \end{aligned}$$

for any $x \in Q(x, 2^{k-v+3})$, where $\frac{1}{\sigma} = \frac{r}{p(\cdot)\tau(\cdot)} + \frac{1}{t(\cdot)}$. The second norm is bounded. The first norm is bounded if and only if

$$\Big\| \frac{\left( g_{v,k,v_P} \right)^{1/r}}{|Q(x, 2^{k-v+3})|^{\tau(\cdot)}} \chi_{Q(x,2^{k-v+3})} \Big\|_{p(\cdot)} \lesssim 1,$$

which follows since $\|\lambda\|_{\mathfrak{f}_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\tau(\cdot)}} \leq 1$. Hence we can apply Lemma 1 to estimate

$$\left( M_{Q(x,2^{k-v+3})}(2^{-kn(r+1/t(\cdot))\tau(\cdot)} g_{v,k,v_P})\,(x) \right)^{\sigma/\tau(x)}$$

by

$$c\; M_{Q(x,2^{k-v+3})} \left( \left( 2^{-kn(r+1/t(\cdot))\tau(\cdot)} g_{v,k,v_P} \right)^{\sigma/\tau(\cdot)} \right)(x) + \min\left( 2^{n(k-v)s}, 1 \right) h\,(x)\,,$$

where $s > 0$ large enough and $h$ is the same function in Lemma 3. Hence the term in (28), with $\sum_{v=0}^{k+3}$ in place of $\sum_{v=0}^{\infty}$ is bounded by

$$\begin{aligned} & c \sum_{v=0}^{k+3} \frac{1}{2^{(w-\frac{n\sigma}{t^{+}})k} + 2^{kns}} \Big\| \frac{M_{Q(\cdot,2^{k-v+3})} \left( g_{v,k,v_P} \right)^{\frac{\sigma}{\tau(\cdot)}}}{|Q(\cdot, 2^{k-v_P+3})|^{r\sigma}} \Big\|_{\frac{p(\cdot)\tau(\cdot)}{r\sigma}} \\ & + \frac{c\,(k+4)^{s}}{2^{(w-\frac{n\sigma}{t^{+}})k} + 2^{kns}}. \end{aligned}$$

Since $\mathcal{M}$ is bounded in $L^{p(\cdot)\tau(\cdot)/r\sigma}$ the last norm is bounded by

$$\Big\|\mathcal{M}\Big(\frac{(g_{v,k,v_P})^{\frac{\sigma}{\tau(\cdot)}}}{|Q(\cdot,2^{k-v_P+3})|^{r\sigma}}\Big)\Big\|_{\frac{p(\cdot)\tau(\cdot)}{r\sigma}} \lesssim \Big\|\frac{(g_{v,k,v_P})^{\frac{\sigma}{\tau(\cdot)}}}{|Q(\cdot,2^{k-v_P+3})|^{r\sigma}}\Big\|_{\frac{p(\cdot)\tau(\cdot)}{r\sigma}}.$$

This term is bounded by 1 if and only if

$$\Big\|\frac{(g_{v,k,v_P})^{1/r}}{|Q(c_P,2^{k-v_P+3})|^{\tau(\cdot)}}\Big\|_{p(\cdot)} \lesssim 1,$$

wich follows since $|Q(\cdot,2^{k-v_P+3})| \geq 1$. Now, with $w > n + c_{\log}(1/q) + c_{\log}(\tau)$, the term in (28), with $\sum_{v=k+4}^{\infty}$ in place of $\sum_{v=0}^{\infty}$ is bounded by

$$\begin{aligned}
&\frac{c\ 2^{(w-\frac{n\sigma}{t^+})k}}{2^{(w-\frac{n\sigma}{t^+})k}+2^{kns}}\Big\|\Big(\sum_{v=k+4}^{\infty}\Big(\frac{\eta_{v,w}*(g_{v,k,v_P})^{\frac{\sigma}{\tau(\cdot)}}}{|Q(\cdot,2^{k-v_P+3})|^{r\sigma}}\Big)^{\frac{q(\cdot)\tau(\cdot)}{r\sigma}}\Big)^{\frac{\sigma r}{q(\cdot)\tau(\cdot)}}\Big\|_{\frac{p(\cdot)\tau(\cdot)}{r\sigma}} \\
&+\frac{c\ 2^{kns}}{2^{(w-\frac{n\sigma}{t^+})k}+2^{kns}} \\
\lesssim\ & 1,
\end{aligned}$$

by Theorem 2. Therefore our estimate (27) follows by taking $0 < s < \frac{w}{n}$ and the fact that

$$d > n(r+1)\tau^+ + n + a + w.$$

*Case 2.* $|P| < 1$. We have

$$|P|^{-\tau(x)} \leq c\,|P|^{-\tau(y)}\,(1+2^{v_P}\,|x-y|)^{c_{\log}(\tau)} \leq c\,|P|^{-\tau(y)}\,(1+2^{v}\,|x-y|)^{c_{\log}(\tau)}$$

for any $x, y \in \mathbb{R}^n$. Hence,

$$\frac{\eta_{v,w}*g_{v,k,v_P}}{|P|^{\tau(\cdot)}} \lesssim \eta_{v,w-c_{\log}(\tau)} * \left(\frac{g_{v,k,v_P}}{|P|^{\tau(\cdot)}}\right).$$

Therefore, we can use the similar arguments above to obtain the desired estimate, where we did not need to use Lemma 1, which could be used only to move $|P|^{\tau(\cdot)}$ inside the convolution and hence the proof is complete.

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